Understanding James' compactness theorem

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Seminario Análisis Funcional-Valencia 2018-19.

Supported by GOBIERNO DE ESPAÑA, MINISTERIO DE ECONOMÍA Y COMPETITIVIDAD, Unión Europea, Fondo Europeo de Desarrollo Regional

Birthday's theorems

- P. Kenderov 2003
- J. Lindenstrauss 2006
- M. Valdivia 2010
- W. Schachermayer 2010
- J. Borwein 2011
- I. Karatzas 2012
- F. Delbaen 2012
- A. Defant 2013
- P. Kenderov 2013
- J. Bonet 2015
- M. Maestre 2015
- D. García 2018

The coauthors

- J. Orihuela and M. Ruiz Galán *A coercive and nonlinear James's weak compactness theorem* Nonlinear Analysis 75 (2012) 598-611.
- J. Orihuela and M. Ruiz Galán *Lebesgue Property for Convex Risk Meausures on Orlicz Spaces* Math. Finan. Econ. 6(1) (2012) 15–35.
- B. Cascales, J. Orihuela and M. Ruiz Gal'an *Compactness, Optimality and Risk* Computational and Analytical Mathematics. Conference in honour of J.M Borwein 60'th birthday. Chapter 10, Springer Verlag 2013, 153–208.

The coauthors

- B. Cascales, J. Orihuela and A. Pérez: *One-sided James Compactness Theorem,* J. Math. Anal. Appli. Volume 445, Issue 2, 1267-1283 (2017).
- J. Orihuela and J. M. Zapata: *Stability in locally $\mathbb{L}^0$-convex modules and a conditional version of James' compactness theorem,* J. Math. Anal. Appl. 452 (2017), no. 2, 1101-1127.
- J. Orihuela: *Conic James' Compactness Theorem,* Journal of Convex Analysis (2018)(3), 1335–1344.
- F. Delbaen and J. Orihuela *Mackey's constraints for James Compactness Theorem and Risk Measures,* (2018) Preprint.

Weak Compactness Theorem of R.C. James

Theorem

A Banach space is reflexive if, and only if, each continuous linear functional attains its supremum on the unit ball

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A bounded and weakly closed subset K of a Banach space is weakly compact if, and only if, each continuous linear functional attains its supremum on K

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Main tool for separable Bnacah spaces

Theorem (Simons)

Let X be a set and $(f_n)_n$ a uniformly bounded sequence in $\ell^\infty(\Gamma)$. If Y is a subset of X such that for every sequence of positive numbers $(\lambda_n)_n$, with $\sum_{n=1}^\infty \lambda_n = 1$, there exists $y \in Y$ such that

$$\sup\{\sum_{n=1}^\infty \lambda_n f_n(y) : x \in X\} = \sum_{n=1}^\infty \lambda_n f_n(y),$$

then we have:

$$\sup_{y\in Y} \limsup_{k\to\infty} f_k(y) = \sup_{x\in X} \limsup_{k\to\infty} f_k(x)$$

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Weak Compactness through Sup–limsup Theorem

Theorem

Let E be a separable Banach space and $K \subset E$ a closed convex and bounded subset. They are equivalent:

1. *K is weakly compact.*
2. *For every sequence $(x_n^*) \subset B_{E^*}$ we have*

$$\sup_{k\in K}\{\limsup_{n\to\infty} x_n^*(k)\} = \sup_{\kappa\in\overline{K}^{w^*}}\{\limsup_{n\to\infty} x_n^*(\kappa)\}$$

Sup–limsup Theorem ⇒ Compactness

- If K is not weakly compact there is $x_0^{**} \in \overline{K}^{w^*} \subset E^{**}$ with $x_0^{**} \notin E$
- The Hahn Banach Theorem provide us $x^{***} \in B_{E^{***}} \cap E^{\perp}$ with $x^{***}(x_0^{**}) = \alpha > 0$
- The separability of E, Ascoli's and Bipolar Theorems permit to construct a sequence $(x_n^*) \subset B_{E^*}$ such that:
  1. $\lim_{n\to\infty} x_n^*(x) = 0$ for all $x \in E$
  2. $x_n^*(x_0^{**}) > \alpha/2$ for all $n \in \mathbb{N}$
- Then

$$0 = \sup_{k\in K}\{\lim_{n\to\infty} x_n^*(k)\} = \sup_{k\in K}\{\limsup_{n\to\infty} x_n^*(k)\} =$$

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Weak Compactness through I-generation

Theorem (Fonf and Lindenstrauss)

Let E be a separable Banach space and $K \subset E$ a closed convex and bounded subset. They are equivalent:

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I-generation ⇒ Weak Compactness

- Take $\{x_n : n \in \mathbb{N}\}$ norm dense in K
- $B_m := \overline{\mathrm{co}(\{x_n : n \leq m\})}^{\|\cdot\|}$ is finite dimensional closed compact set
- $D_m := B_m + \delta B_{E^{**}}$ for $\delta > 0$ fixed
- Since $K \subset \bigcup_{m=1}^{\infty} D_m$, the I-generation says that

$$\overline{\bigcup_{m}^{\infty} \overline{D_m}^{w^*}}^{\|\cdot\|} = \overline{K}^{w^*}.$$

- So $(\bigcup_m^\infty B_m) + 2\delta B_{E^{**}} \supset \overline{K}^{w^*}$.
- Finally $K = \overline{K}^{\|\cdot\|} = \bigcap_{\delta>0}(\bigcup_m^\infty B_m) + 2\delta B_{E^{**}} = \overline{K}^{w^*}$.

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Fonf-Lindenstrauss = Simons

Theorem (Cascales, Fonf, Troyanski and Orihuela, J.F.A.-2010)

Let E be a Banach space, $K \subset E^*$ be w^*-compact convex, $B \subset K$, TFAE:

1. *For any covering $B \subset \cup_{n=1}^{\infty} D_n$ by an increasing sequence of convex subsets $D_n \subset K$, we have*

$$\overline{\cup_n^{\infty} \overline{D_n}^{w^*}}^{\|\cdot\|} = K.$$

2. $\sup_{f \in B} (\limsup_k f(x_k)) = \sup_{g \in K} (\limsup_k g(x_k))$ *for every sequence $\{x_k\} \subset B_X$.*
3. $\sup_{f \in B} (\limsup_k f(x_k)) \geq \inf_{\sum \lambda_i = 1, \lambda_i \geq 0} (\sup_{g \in K} g(\sum \lambda_i x_i))$ *for every sequence $\{x_k\} \subset B_X$.*

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Lemma

If $\{f_n\}_{n\geq 1}$ *is a pointwise bounded sequence in* $\mathbb{R}^X$ *and* $\varepsilon > 0$*, then for every* $m \geq 1$ *there exists* $g_m \in \mathrm{co}_{\sigma_p}\{f_n \colon n \geq m\}$ *such that*

$$S_X\left(\sum_{n=1}^{m-1} \frac{g_n}{2^n}\right) \leq \left(1 - \frac{1}{2^{m-1}}\right) S_X\left(\sum_{n=1}^{\infty} \frac{g_n}{2^n}\right) + \frac{\varepsilon}{2^{m-1}}.$$

- inductively, for each $m \geq 1$, $g_m \in \mathrm{co}_{\sigma_p}\{f_n \colon n \geq m\}$ satisfying $S_X\left(\sum_{n=1}^{m-1} \frac{g_n}{2^n} + \frac{g_m}{2^{m-1}}\right) \leq \inf_{g \in \mathrm{co}_{\sigma_p}\{f_n \colon n \geq m\}} S_X\left(\sum_{n=1}^{m-1} \frac{g_n}{2^n} + \frac{g}{2^{m-1}}\right) + \frac{2\varepsilon}{4^m}$.
- The existence of such g_m follows from the easy fact that $\inf_{g \in \mathrm{co}_{\sigma_p}\{f_n \colon n \geq m\}} S_X(g) > -\infty$, according with the pointwise boundeness of our sequence $\{f_n\}_{n \geq 1}$.
- Since $2^{m-1} \sum_{n=m}^{\infty} \frac{g_n}{2^n} \in \mathrm{co}_{\sigma_p}\{f_n \colon n \geq m\}$, then $S_X\left(\left(\sum_{n=1}^{m-1} \frac{g_n}{2^n}\right) + \frac{g_m}{2^{m-1}}\right) \leq S_X\left(\sum_{n=1}^{\infty} \frac{g_n}{2^n}\right) + \frac{2\varepsilon}{4^m}$.

Simons' inequality for pointwise bounded subsets

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- From the equality
  $\sum_{n=1}^{m-1} \frac{g_n}{2^n} = \sum_{k=1}^{m-1} \frac{1}{2^{m-k}} \left(\left(\sum_{n=1}^{k-1} \frac{g_n}{2^n} \right) + \frac{g_k}{2^{k-1}} \right),$
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S_X\left(\sum_{n=1}^{m-1} \frac{g_n}{2^n}\right) &\leq \sum_{k=1}^{m-1} \frac{1}{2^{m-k}} S_X\left(\left(\sum_{n=1}^{k-1} \frac{g_n}{2^n}\right) + \frac{g_k}{2^{k-1}}\right) \\
&\leq \sum_{k=1}^{m-1} \frac{1}{2^{m-k}} \left(S_X\left(\sum_{n=1}^{\infty} \frac{g_n}{2^n}\right) + \frac{2\varepsilon}{4^k}\right) \\
&= \left(1 - \frac{1}{2^{m-1}}\right) S_X\left(\sum_{n=1}^{\infty} \frac{g_n}{2^n}\right) + \left(1 - \frac{1}{2^{m-1}}\right) \frac{2\varepsilon}{2^{m}} \\
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Simons' inequality for pointwise bounded subsets

- From the equality $\sum_{n=1}^{m-1} \frac{g_n}{2^n} = \sum_{k=1}^{m-1} \frac{1}{2^{m-k}} \left(\left(\sum_{n=1}^{k-1} \frac{g_n}{2^n} \right) + \frac{g_k}{2^{k-1}} \right)$,
- $$\begin{aligned} S_X\left(\sum_{n=1}^{m-1} \frac{g_n}{2^n}\right) &\leq \sum_{k=1}^{m-1} \frac{1}{2^{m-k}} S_X\left(\left(\sum_{n=1}^{k-1} \frac{g_n}{2^n}\right) + \frac{g_k}{2^{k-1}}\right) \\ &\leq \sum_{k=1}^{m-1} \frac{1}{2^{m-k}} \left(S_X\left(\sum_{n=1}^{\infty} \frac{g_n}{2^n}\right) + \frac{2\varepsilon}{4^k} \right) \\ &= \left(1 - \frac{1}{2^{m-1}}\right) S_X\left(\sum_{n=1}^{\infty} \frac{g_n}{2^n}\right) + \left(1 - \frac{1}{2^{m-1}}\right) \frac{2\varepsilon}{2^m} \\ &\leq \left(1 - \frac{1}{2^{m-1}}\right) S_X\left(\sum_{n=1}^{\infty} \frac{g_n}{2^n}\right) + \frac{\varepsilon}{2^{m-1}}, \end{aligned}$$

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-
$$
\begin{aligned}
S_X\left(\sum_{n=1}^{m-1} \frac{g_n}{2^n}\right) &\leq \sum_{k=1}^{m-1} \frac{1}{2^{m-k}} S_X\left(\left(\sum_{n=1}^{k-1} \frac{g_n}{2^n}\right) + \frac{g_k}{2^{k-1}}\right) \\
&\leq \sum_{k=1}^{m-1} \frac{1}{2^{m-k}} \left(S_X\left(\sum_{n=1}^{\infty} \frac{g_n}{2^n}\right) + \frac{2\varepsilon}{4^k}\right) \\
&= \left(1 - \frac{1}{2^{m-1}}\right) S_X\left(\sum_{n=1}^{\infty} \frac{g_n}{2^n}\right) + \left(1 - \frac{1}{2^{m-1}}\right) \frac{2\varepsilon}{2^m} \\
&\leq \left(1 - \frac{1}{2^{m-1}}\right) S_X\left(\sum_{n=1}^{\infty} \frac{g_n}{2^n}\right) + \frac{\varepsilon}{2^{m-1}},
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Theorem (Simons' inequality in $\mathbb{R}^X$)

Let X be a nonempty set, let $\{f_n\}_{n\geq 1}$ be a pointwise bounded sequence in $\mathbb{R}^X$ and let Y be a subset of X such that for every $g \in \mathrm{co}_{\sigma_p}\{f_n \colon n \geq 1\}$ there exists

$$y \in Y \text{ with } g(y) = S_X(g).$$

Then

$$\inf_{g \in \mathrm{co}_{\sigma_p}\{f_n \colon n \geq 1\}} S_X(g) \leq S_Y\left(\limsup_n f_n\right).$$

Simons' inequality for pointwise bounded subsets

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Proof of pointwise bounded Simons' inequality

- For $\varepsilon > 0$ we are going to find $y \in Y$ and $g \in \mathrm{co}_{\sigma_p}\{f_n \colon n \geq 1\}$ such that $S_X(g) - \varepsilon \leq \limsup_n f_n(y)$.
- Fix $\varepsilon > 0$. The former Lemma provides us with a sequence $\{g_m\}_{m\geq 1}$ in $\mathbb{R}^X$ such that for every $m \geq 1$, $g_m \in \mathrm{co}_{\sigma_p}\{f_n \colon n \geq m\}$ and

$$S_X\left(\sum_{n=1}^{m-1} \frac{g_n}{2^n}\right) \leq \left(1 - \frac{1}{2^{m-1}}\right) S_X\left(\sum_{n=1}^{\infty} \frac{g_n}{2^n}\right) + \frac{\varepsilon}{2^{m-1}}. \quad (1)$$

- $g := \sum_{n=1}^{\infty} \frac{g_n}{2^n} \in \mathrm{co}_{\sigma_p}\{f_n \colon n \geq 1\}$, then there exists $y \in Y$ with

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Proof of pointwise bounded Simons' inequality II

- So given $m \geq 1$, it follows from (1) and (2):

$$\left(1 - \frac{1}{2^{m-1}}\right) g(y) + \frac{\varepsilon}{2^{m-1}} \geq S_X\left(\sum_{n=1}^{m-1} \frac{g_n}{2^n}\right) \geq \sum_{n=1}^{m-1} \frac{g_n(y)}{2^n} = g(y) - \sum_{n=m}^{\infty} \frac{g_n(y)}{2^n}.$$

- Therefore,

$$\inf_{m \geq 1} 2^{m-1} \sum_{n=m}^{\infty} \frac{g_n(y)}{2^n} \geq g(y) - \varepsilon. \qquad (3)$$

- Since for every $m \geq 1$ we have $2^{m-1} \sum_{n=m}^{\infty} 2^n = 1$, we conclude that

$$\sup_{n \geq m} f_n(y) \geq 2^{m-1} \sum_{n=m}^{\infty} \frac{g_n(y)}{2^n}.$$

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- Now, with this last inequality in mind:

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- together with (2) and (3) we arrive at

$$\begin{aligned}
\limsup_n f_n(y) &= \inf_{m\geq 1} \sup_{n\geq m} f_n(y) \\
&\geq \inf_{m\geq 1} 2^{m-1} \sum_{n=m}^{\infty} \frac{g_n(y)}{2^n} \\
&\geq g(y) - \varepsilon \\
&= S_X(g) - \varepsilon,
\end{aligned}$$

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Sup-limsup Theorem

Corollary (Simons' sup-limsup theorem in $\mathbb{R}^X$)

Let X be a nonempty set, let $\{f_n\}_{n\geq 1}$ be a pointwise bounded sequence in $\mathbb{R}^X$ and let Y be a subset of X such that for every $g \in \mathrm{co}_{\sigma_p}\{f_n : n \geq 1\}$ there exits

$$y \in Y \text{ with } g(y) = S_X(g).$$

Then

$$S_X\left(\limsup_n f_n\right) = S_Y\left(\limsup_n f_n\right).$$

Corollary (Simons' sup-limsup theorem in $\mathbb{R}^{\mathbf{X}}$)

Let X be a nonempty set, let $\{f_n\}_{n\geq 1}$ be a pointwise bounded sequence in $\mathbb{R}^X$ and let Y be a subset of X such that for every $g \in \mathrm{co}_{\sigma_p}\{f_n \colon n \geq 1\}$ there exits

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A proof for Sup-limsup theorem

- Let us assume, arguing by *reductio ad absurdum*, that there exists $x_0 \in X$ such that
$$\limsup_n f_n(x_0) > S_Y\left(\limsup_n f_n\right).$$

- We assume then, passing to a subsequence if necessary, that
$$\inf_{n\geq 1} f_n(x_0) > S_Y\left(\limsup_n f_n\right).$$

- In particular,
$$\inf_{g\in \mathrm{co}_{\sigma_p}\{f_n:\ n\geq 1\}} g(x_0) > S_Y\left(\limsup_n f_n\right),$$

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and then, by applying Simons' inequality, we arrive at

$$
\begin{aligned}
S_Y\left(\limsup_n f_n\right) &\geq \inf_{g\in \mathrm{co}_{\sigma p}\{f_n:\ n\geq 1\}} S_X(g) \\
&\geq \inf_{g\in \mathrm{co}_{\sigma p}\{f_n:\ n\geq 1\}} g(x_0) \\
&> S_Y\left(\limsup_n f_n\right),
\end{aligned}
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a contradiction.

A proof for Sup-limsup theorem

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$$\begin{aligned} S_Y\left(\limsup_n f_n\right) &\geq \inf_{g \in \mathrm{co}_{\sigma p}\{f_n \colon n \geq 1\}} S_X(g) \\ &\geq \inf_{g \in \mathrm{co}_{\sigma p}\{f_n \colon n \geq 1\}} g(x_0) \\ &> S_Y\left(\limsup_n f_n\right), \end{aligned}$$

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Theorem (Unbounded Rainwater–Simons' theorem)

If E is a Banach space, C is a subset of E^*, B is a nonempty subset of C and $\{x_n\}_{n\geq 1}$ is a bounded sequence in E such that for every $x \in \mathrm{co}_\sigma\{x_n : \ n \geq 1\}$ there exists

$$b^* \in B \text{ with } b^*(x) = S_C(x),$$

then

$$S_B\left(\limsup_n x_n\right) = S_C\left(\limsup_n x_n\right).$$

As a consequence

$$\sigma(E,B)\text{-}\lim_n x_n = 0 \ \Rightarrow \ \sigma(E,C)\text{-}\lim_n x_n = 0.$$

whenever $x_n(C) \geq 0$ for every $n \in \mathbb{N}$, or $C = -C$.

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Theorem (Fonf–Lindenstrauss' theorem)

Let E be a Banach space, B a bounded subset of E^* such that for every $x \in E$ there exists some $b_0^* \in B$ satisfying $b_0^*(x) = \sup_{b^* \in B} b^*(x)$. Then we have that, for every covering $B \subset \bigcup_{n=1}^{\infty} D_n$ by an increasing sequence of w^*-closed convex subsets $D_n \subset \overline{\mathrm{co}(B)}^{w^*}$, the following equality holds true

$$\overline{\cup_{n=1}^{\infty} D_n}^{\|\cdot\|} = \overline{\mathrm{co}(B)}^{w^*}. \tag{4}$$

A proof for the I-formula

- If $z_0^* \in \overline{\mathrm{co}(B)}^{w^*}$ such that $z_0^* \notin \overline{\cup_{n=1}^{\infty} D_n}^{\|\cdot\|}$. Fix $\delta > 0$ such that
$$B[z_0^*, \delta] \cap D_n = \emptyset, \text{ for every } n \geq 1. \tag{5}$$

- The separation theorem in (E^*, w^*), when applied to the w^*-compact set $B[0, \delta]$ and the w^*-closed set $D_n - z_0^*$, provides us with a norm-one $x_n \in E$ and $\alpha_n \in \mathbb{R}$ such that
$$\inf_{v^* \in B[0,\delta]} x_n(v^*) > \alpha_n > \sup_{y^* \in D_n} x_n(y^*) - x_n(z_0^*).$$
But
$$-\delta = \inf_{v^* \in B[0,\delta]} x_n(v^*),$$

- and consequently the sequence $\{x_n\}_{n \geq 1}$ in B_E satisfies
$$x_n(z_0^*) - \delta > x_n(y^*) \tag{6}$$
for each $n \geq 1$ and $y^* \in D_n$.

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A proof for the I-formula II

- Fix a w^*-cluster point $x^{**} \in B_{E^{**}}$ of the sequence $\{x_n\}_{n\geq 1}$ and let $\{x_{n_k}\}_{k\geq 1}$ be a subsequence of $\{x_n\}_{n\geq 1}$ such that $x^{**}(z_0^*) = \lim_k x_{n_k}(z_0^*)$.
- We can and do assume that for every $k \geq 1$,

$$x_{n_k}(z_0^*) > x^{**}(z_0^*) - \frac{\delta}{2}. \tag{7}$$

- Since $B \subset \cup_{n=1}^{\infty} D_n$ and $\{D_n\}_{n\geq 1}$ is an increasing sequence of sets, given $b^* \in B$ there exists $k_0 \geq 1$ such that $b^* \in D_{n_k}$ for each $k \geq k_0$.

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- Now inequality (6) yields

$$x^{**}(z_0^*) - \delta \geq \limsup_k x_{n_k}(b^*), \quad \text{for every } b^* \in B, \qquad (8)$$

- and, on the other hand, inequality (7) implies that

$$w(z_0^*) \geq x^{**}(z_0^*) - \frac{\delta}{2}, \quad \text{for every } w \in \mathrm{co}_\sigma\{x_{n_k} : k \geq 1\}. \quad (9)$$

- Now Simons' inequality can be applied to the sequence $\{x_{n_k}\}_{k\geq 1}$, to deduce

$$x^{**}(z_0^*) - \delta \overset{(8)}{\geq} \sup_{b^* \in B} \limsup_k x_{n_k}(b^*) \geq$$

$$\inf\left\{ \sup\{w(z^*) : z^* \in \overline{\mathrm{co}(B)}^{w^*}, w \in \mathrm{co}_\sigma\{x_{n_k} : k \in \mathbb{N}\}\right\}$$

$$\geq \inf\left\{ w(z_0^*) : w \in \mathrm{co}_\sigma\{x_{n_k} : k \in \mathbb{N}\}\right\}$$

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A proof for the I-formula III

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$$\overset{(9)}{\geq} x^{**}(z_0^*) - \frac{\delta}{2}.$$

A proof for the I-formula III

- Now inequality (6) yields
$$x^{**}(z_0^*) - \delta \geq \limsup_k x_{n_k}(b^*), \quad \text{for every } b^* \in B, \tag{8}$$
- and, on the other hand, inequality (7) implies that
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Some questions

Cascales, Fonf, Troyanski and myself (JFA 2012) proved the equivalence between Simons' inequality, the sup-limsup theorem, and the (I)-formula of Fonf and Lindenstrauss in the bounded case. We ask the following exercise:

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We have seen the implications:
Simons' inequality $\to$ *Sup-limsup theorem*
Simons' inequality $\to$ *I- formula, and both of them give us the proof of James theorem for separable Banach spaces. Could you complete the proof of the equivalence of the three tools?*

In the unbounded case we propose the following question:

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Are the unbounded versions of Simons' inequality and the unbounded sup-limsup theorem equivalent to some kind of I-formula for the unbounded case?

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Our answer

Theorem (B. Cascales, J. Orihuela and A. Pérez)

Let $D \subset E^*$ be a weak-compact convex set with $0 \notin D$, and let $C \subset E^*$ be a weak*-closed convex set such that every $x \in L_D$, where $L_D = \{x \in E : x(D) < 0\}$, has finite supremum on C. Given $B \subset C$ the following assertions are equivalent:*

(i) *For every $B \subseteq \bigcup_{n=1}^{\infty} K_n$ with an increasing sequence of weak*-compact convex subsets $K_n \subset C$ we have that $C \subset \overline{\bigcup_{n=1}^{\infty} K_n + \Lambda_D}^{\|\cdot\|}$ where Λ_D is the cone generated by D.*

(ii) *For every bounded sequence $(x_n)_{n\in\mathbb{N}}$ in L_D:*

$$\sup_{b^*\in B}\left(\limsup_n \langle b^*, x_n\rangle\right) = \sup_{c^*\in C}\left(\limsup_n \langle c^*, x_n\rangle\right).$$

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Going to James' theorem for arbitrary Banach spaces

Lemma

Suppose that E is a Banach space, $\{x_n\}_{n\geq 1}$ is a bounded sequence in E and x_0^{**} in E^{**} is a w^*-cluster point of $\{x_n\}_{n\geq 1}$ with $d(x_0^{**}, E) > 0$. Then for every α with $d(x_0^{**}, E) > \alpha > 0$ there exists a sequence $\{x_n^*\}_{n\geq 1}$ in B_{E^*} such that

$$\langle x_n^*, x_0^{**}\rangle > \alpha \tag{10}$$

whenever $n \geq 1$, and

$$\langle x_0^*, x_0^{**}\rangle = 0 \tag{11}$$

for any $x_0^* \in L\{x_n^*\}$.

We are denoting with $L\{x_n^*\}$ the non void set of w^*-cluster points of the sequence (x_n^*)

A proof for the lemma

- The Hahn–Banach theorem applies to provide us with $x^{***} \in B_{E^{***}}$ satisfying $x^{***}_{|E} = 0$ and $x^{***}(x_0^{**}) = d(x_0^{**}, E)$.
- For every $n \geq 1$ the set $V_n := \{y^{***} \in E^{***} : y^{***}(x_0^{**}) > \alpha,\ |y^{***}(x_i)| \leq \frac{1}{n}, i \leq n\}$ is a w^*-open neighborhood of x^{***}, and therefore, by Goldstein's theorem, we can pick up $x_n^* \in B_{E^*} \cap V_n$.
- The sequence $\{x_n^*\}_{n\geq 1}$ clearly satisfies $\lim_n \langle x_n^*, x_p \rangle = 0, \quad$ for all $p \in \mathbb{N}$, and for each $n \geq 1$, $\langle x_n^*, x_0^{**} \rangle > \alpha$.
- Fix an arbitrary $x_0^* \in L\{x_n^*\}$. For every $p \geq 1$ we have that $\langle x_0^*, x_p \rangle = 0$, and thus $\langle x_0^*, x_0^{**} \rangle = 0$, because $x_0^{**} \in \overline{\{x_p : p = 1, 2, \cdots\}}^{w^*}$.

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James' theorem for spaces with w^*-sequential compact dual ball

Theorem

Let E be a Banach space with a w^*-convex block compact dual unit ball. If a bounded subset A of E is not weakly relatively compact, then there exists a sequence of linear functionals $\{y_n^*\}_{n\geq 1} \subset B_{E^*}$ with a w^*-limit point y_0^*, and some $g^* \in \mathrm{co}_\sigma\{y_n^* : n \geq 1\}$, such that $g^* - y_0^*$ doest not attain its supremum on A.

A first proof

- The Eberlein–Šmulian theorem provide us a sequence $\{x_n\}_{n\geq 1}$ in A and a w^*-cluster point $x_0^{**} \in E^{**} \setminus E$ of it. Former Lemma applies to provide us with a sequence $\{x_n^*\}_{n\geq 1}$ in B_{E^*} and $\alpha > 0$ satisfying (10) and (11).
- Let $\{y_n^*\}_{n\geq 1}$ be a convex-block sequence of $\{x_n^*\}_{n\geq 1}$ and let $y_0^* \in B_{E^*}$ such that $w^*\text{-}\lim_n y_n^* = y_0^*$. It is clear that (10) and (11) are valid when replacing $\{x_n^*\}_{n\geq 1}$ and x_0^* with $\{y_n^*\}_{n\geq 1}$ and y_0^*, respectively.
- Then

$$\begin{aligned} S_{\overline{A}^{w^*}}\left(\limsup_n (y_n^* - y_0^*)\right) &\geq \limsup_n (y_n^* - y_0^*)(x_0^{**}) \\ &\geq \alpha > 0 \\ &= S_A\left(\limsup_n (y_n^* - y_0^*)\right), \end{aligned}$$

- so in view of the Suplim-Sup theorem of Simons, there exists $g^* \in \mathrm{co}_\sigma\{y_n^* : n \geq 1\}$ such that $g^* - y_0^*$ does not attain its supremum on A, as announced.

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Theorem

Let A be a bounded subset of a Banach space E. Then A is weakly relatively compact if, and only if, for every bounded sequence $\{x_n^*\}_{n\geq 1}$ in E^* we have

$$\operatorname{dist}_{\|\cdot\|_A}(L\{x_n^*\}, \operatorname{co}\{x_n^* : n \geq 1\}) = 0. \tag{12}$$

- We first prove that if A is weakly relatively compact then equality (12) holds for any bounded sequence $\{x_n^*\}_{n\geq 1}$ in E^*.
- Since $\overline{\mathrm{co}(A)}^{\|\cdot\|}$ is weakly compact by the Krein–Šmulian theorem, the seminorm $\|\cdot\|_A = \|\cdot\|_{\overline{\mathrm{co}(A)}^{\|\cdot\|}}$ is continuous for the Mackey topology $\tau(E^*, E)$.
- Hence we have the inclusions $L\{x_n^*\} \subset \overline{\mathrm{co}\{x_n^* : n \geq 1\}}^{w^*} = \overline{\mathrm{co}\{x_n^* : n \geq 1\}}^{\tau(E^*,E)} \subset \overline{\mathrm{co}\{x_n^* : n \geq 1\}}^{\|\cdot\|_A}$, that clearly explain the validity of (12).

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The reverse impliction

- Let us assume that A is not relatively weakly compact in E. Then the Eberlein–Šmulian theorem guarantees the existence of a sequence $\{x_n\}_{n\geq 1}$ in A with a w^*-cluster point $x_0^{**} \in E^{**} \setminus E$.
- If $d(x_0^{**}, E) > \alpha > 0$, an appeal to our Lemma provides us with a sequence $\{x_n^*\}_{n\geq 1}$ in B_{E^*} satisfying

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- Therefore we have that

$$\|\sum_{i=1}^{n} \lambda_i x_{n_i}^* - x_0^*\|_A \geq \left\langle \sum_{i=1}^{n} \lambda_i x_{n_i}^* - x_0^*, x_0^{**} \right\rangle > \alpha$$

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Connection with Pryce's arguments for the general case

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Let E be a Banach space, A a bounded subset of E with $A = -A$, $\{x_n^*\}_{n\geq 1}$ a bounded sequence in the dual space E^*, and D its norm-closed linear span in E^*. Then there exists a subsequence $\{x_{n_k}^*\}_{k\geq 1}$ of $\{x_n^*\}_{n\geq 1}$ such that

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$$S_A\left(x^* - \liminf_k x_{n_k}^*\right) = S_A\left(x^* - \limsup_k x_{n_k}^*\right) =$$

$$= \mathrm{dist}_{\|\cdot\|_A}(x^*, L\{x_{n_k}^*\})$$

for all $x^* \in D$.

Connection with Pryce' s arguments for the general case

- Pryce argument implies the existence of a subsequence $\{x^*_{n_k}\}_{k\geq 1}$ of $\{x^*_n\}_{n\geq 1}$ such that $S_A\left(x^* - \liminf_k x^*_{n_k}\right) = S_A\left(x^* - \limsup_k x^*_{n_k}\right)$ for all $x^* \in D$.
- Since for any $h^* \in L\{x^*_{n_k}\}$ we have $\liminf_k x^*_{n_k}(a) \leq h^*(a) \leq \limsup_k x^*_{n_k}(a)$ for all $a \in A$,
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Theorem (James-Pryce undetermined function technique)

Let X be a nonempty set, $\{h_j\}_{j\geq 1}$ a bounded sequence in $\ell^\infty(X)$, and $\delta > 0$ such that

$$S_X\left(h - \limsup_j h_j\right) = S_X\left(h - \liminf_j h_j\right) \geq \delta,$$

whenever $h \in \mathrm{co}_\sigma\{h_j : j \geq 1\}$. Then there exists a sequence $\{g_i\}_{i\geq 1}$ in $\ell^\infty(X)$ with

$$g_i \in co_\sigma\{h_j : j \geq i\}, \quad \text{for all } i \geq 1,$$

and there exists $g_0 \in \mathrm{co}_\sigma\{g_i : i \geq 1\}$ such that for all $g \in \ell^\infty(X)$ with

$$\liminf_i g_i \leq g \leq \limsup_i g_i \quad \text{on } X,$$

the function $g_0 - g$ doest not attain its supremum on X .

Theorem (**James**)

Let A be a nonempty bounded subset of a Banach space E which is not weakly relatively compact. Then there exists a sequence $\{g_n^*\}_{n\geq 1}$ in B_{E^*} and some $g_0^* \in \mathrm{co}_\sigma\{g_n^* : \ n \geq 1\}$ such that, for every $h \in \ell^\infty(A)$ with

$$\liminf_n g_n^* \leq h \leq \limsup_n g_n^* \quad \text{on } A,$$

we have that $g_0^* - h$ does not attain its supremum on A.

- Without loss of generality we can assume that A is convex and that $A = -A$.
- Our Theorem above gives us a sequence $\{x_n^*\}_{n\geq 1}$ in B_{E^*} such that $\mathrm{dist}_{\|\cdot\|_A}(L\{x_n^*\}, \mathrm{co}\{x_n^* : n \geq 1\}) > 0$.
- By the former theorem there exists a subsequence $\{x_{n_k}^*\}_{k\geq 1}$ of $\{x_n^*\}_{n\geq 1}$ that verifies the hypothesis there.
- So we find a sequence $\{g_n^*\}_{n\geq 1}$ with $g_n^* \in \mathrm{co}_\sigma\{x_{n_k}^* : \ k \geq n\}$, for every $n \in \mathbb{N}$, and $g_0^* \in \mathrm{co}_\sigma\{g_n^* : \ n \geq 1\}$ such that $g_0^* - h$ doest not attain its supremum on A, where h is any function in $\ell^\infty(A)$ with $\liminf_n g_n^* \leq h \leq \limsup_n g_n^*$ on A.
- In particulas for every cluster $g^* \in L\{g_n^*\}$ we see that $g_0^* - g^*$ doest not attain its supremum on A

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The Theorem of James as a minimization problem

- Let us fix a Banach space E with dual E^*
- K is a closed convex set in the Banach space E
- $\iota_K(x) = 0$ if $x \in K$ and $+\infty$ otherwise
- $x^* \in E^*$ attains its supremum on K at $x_0 \in K \Leftrightarrow \iota_K(y) - \iota_K(x_0) \geq x^*(y - x_0)$ for all $y \in E$
- The minimization problem

$$\min\{\iota_K(\cdot) - x^*(\cdot)\}$$

  on E for every $x^* \in E^*$ has always solution if and only if the set K is weakly compact
- When the minimization problem

$$\min\{\alpha(\cdot) + x^*(\cdot)\}$$

  on E has solution for all $x^* \in E^*$ and a fixed proper function $\alpha : E \to (-\infty, +\infty]$?

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Theorem (J. Orihuela and M. Ruiz)

Let E be a Banach space, $\alpha : E \to (-\infty, +\infty]$ proper, (lower semicontinuous) function with

$$\lim_{\|x\|\to\infty} \frac{\alpha(x)}{\|x\|} = +\infty$$

Suppose that there is $c \in \mathbb{R}$ such that the level set $\{\alpha \leq c\}$ fails to be (relatively) weakly compact. Then there is $x^* \in E^*$ such that,the infimum

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Maximizing $\{x^*(x) - \alpha(x) : x \in E\}$

Theorem (J. Orihuela and M. Ruiz. – J. Saint Raymond)

Let E be a Banach space, $\alpha : E \to (-\infty, +\infty]$ proper, lower semicontinuous function, then we have:

- *If $\partial\alpha(E) = E^*$ then the level sets $\{\alpha \leq c\}$ are weakly compact for all $c \in \mathbb{R}$.*
- *If α has weakly compact level sets and the Fenchel-Legendre conjugate α^* is finite, i.e. $\sup\{x^*(x) - \alpha(x) : x \in E\} < +\infty$ for all $x^* \in E^*$, then $\partial\alpha(E) = E^*$*

Theorem (Jouini-Schachermayer-Touzi)

Let $U : \mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ be a monetary utility function with the Fatou property and $U^* : \mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})^* \to [0, \infty]$ its Fenchel-Legendre transform. They are equivalent:

1. $\{U^* \leq c\}$ *is* $\sigma(\mathbb{L}^1, \mathbb{L}^\infty)$*-compact subset for all* $c \in \mathbb{R}$
2. *For every* $X \in \mathbb{L}^\infty$ *the infimum in the equality*
$$U(X) = \inf_{Y \in \mathbb{L}^1} \{U^*(Y) + \mathbb{E}[XY]\},$$
is attained
3. *For every uniformly bounded sequence* (X_n) *tending a.s. to* X *we have*
$$\lim_{n \to \infty} U(X_n) = U(X).$$

Order Continuity of Risk Measures

Theorem (J. Orihuela and M. Ruiz- Lebesgue Risk Measures on Orlicz spaces)

Let $\rho(X) = \sup_{Y \in \mathbb{M}^{\Psi^*}} \{\mathbb{E}_{\mathbb{P}}[-XY] - \alpha(Y)\}$ be a finite convex risk measure on L^{Ψ} with $\alpha : (\mathbb{L}^{\Psi}(\Omega, \mathcal{F}, \mathbb{P})^* \to (-\infty, +\infty]$ a penalty function w^*-lower semicontinuos. T.F.A.E.:

(i) *For all $c \in \mathbb{R}$, $\alpha^{-1}((-\infty, c])$ is a relatively weakly compact subset of $\mathbb{M}^{\Psi^*}(\Omega, \mathcal{F}, \mathbb{P})$.*

(ii) *For every $X \in \mathbb{L}^{\Psi}(\Omega, \mathcal{F}, \mathbb{P})$, the supremum in the equality*

$$\rho(X) = \sup_{Y \in \mathbb{M}^{\Psi^*}} \{\mathbb{E}_{\mathbb{P}}[-XY] - \alpha(Y)\}$$

is attained.

(iii) *ρ is sequentially order continuous*

Theorem (Reflexivity frame)

Let E be a real Banach space and

$$\alpha : E \longrightarrow \mathbb{R} \cup \{+\infty\}$$

a a function such that $\mathrm{dom}(\alpha)$ has nonempty interior and for all $x^* \in E^*$ there exists $x_0 \in E$ with

$$\alpha(x_0) + x^*(x_0) = \inf_{x \in E} \{\alpha(x) + x^*(x)\}$$

Then E is reflexive.

- Fix an open ball $B \subseteq \mathrm{dom}(\alpha)$
- $B = \bigcup_{p=1}^{+\infty} B \cap \overline{\alpha^{-1}((-\infty, p])}^{\sigma(E,E^*)}$
- Baire Category Theorem $\Rightarrow$ there is $q \in \mathbb{N}$:

$$B \cap \overline{\alpha^{-1}((-\infty, q])}^{\sigma(E,E^*)}$$

  has non void interior relative to B
- There is G open in E such that $\emptyset \neq B \cap G \subset B \cap \overline{\alpha^{-1}((-\infty, q])}^{\sigma(E,E^*)}$
- $\overline{\alpha^{-1}((-\inf, q])}^{\sigma(E,E^*)}$ weakly compact $\Rightarrow$ G contains an open relatively weakly compact ball
- B_E is weakly compact

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F. Delbaen problem

Let C be a convex, bounded and closed, but not weakly compact subset of the Banach space E with $0 \notin C$. The following problem has been posed by F. Delbaen motivated by risk measures theory:

Question

Let $E = \mathbb{L}^1(\Omega, \mathcal{F}, \mathbb{P})$. Is it possible to find a linear functional not attaining its minimum on C and that stays strictly positive on C?

Theorem (Birthday Theorem for A. Defant 2013)

Let E be a separable Banach space. Let C be a closed, convex and bounded subset of $E \setminus \{0\}$, $D \subset C$ a relatively weakly compact set of directions such that, for every $x^* \in E^*$, we have that $\inf\{x^*(c) : c \in C\}$ is attained at some point of C whenever $x^*(d) > 0$ for every $d \in D$. Then C is weakly compact.

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Solution to Delbaen's question

Theorem (Birthday Theorem for Pepe Bonet 2015)

Let H be a uniformly integrable subset of $\mathbb{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ with $0 \notin \overline{\text{co}(H)}$. Suppose that A is a subset of $\mathbb{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ such that every $Y \in \mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})$ with $\inf\{\mathbb{E}[X \cdot Y] : X \in H\} > 0$ satisfies that $\inf\{\mathbb{E}[X \cdot Z] : Z \in A\}$ is attained. Then A is uniformly integrable.

Theorem (B. Cascales, J. Orihuela and A. Pérez – W. Moors)

Let E be a Banach space. Let A and B be bounded, closed and convex sets with distance $d(A, B) > 0$. If every $x^* \in E^*$ with

$$\sup(x^*, B) < \inf(x^*, A)$$

attains its infimum on A and its supremum on B, then A and B are both weakly compact.

Theorem (Birthday Theorem for Domingo 2018)

Let E be a Banach space and D be a weakly compact convex subset of E with $0 \notin D$. If A is a bounded subset of E such that every $x^* \in E^*$ with $x^*(D) > 0$ attains its supremum on A, then A is weakly relatively compact.

Theorem (One-sided undefined function technique)

Let A be a convex bounded not relatively weakly compact subset of a Banach space E. Let us fix a convex weakly compact subset D of E which does not contain the origin. Then there is a sequence $\{x_n^*\}_{n\geq 1}$ in B_{E^*} and $g_0^* \in co_\sigma\{x_n^* : n \geq 1\}$ such that for all $h \in \ell_\infty(A)$ satisfying that for all $a \in A$,

$$\liminf_{n\geq 1} x_n^*(a) \leq h(a) \leq \limsup_{n\geq 1} x_n^*(a),$$

we have that

$$g_0^* - h \text{ does not attain its supremum on } A.$$

and $(g_0^* - h)(d) > 0$ for every $d \in D$.

The tool

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Theorem (Second Birthday Theorem for Domingo 2018)

Let A be a convex bounded subset of a Banach space E. Let us fix a convex weakly compact subset D of E which does not contain the origin, a functional $z_0^* \in E^*$ and $\epsilon > 0$. Then there is a bounded sequence $\{x_n^*\}_{n\geq 1}$ in E^* and $g_0^* \in co_\sigma\{x_n^* : n \geq 1\}$ such that for all $h \in \ell_\infty(A)$ satisfying that for all $a \in A$,

$$\liminf_{n\geq 1} x_n^*(a) \leq h(a) \leq \limsup_{n\geq 1} x_n^*(a),$$

we have that

$$z_0^* + h - g_0^* \text{ does not attain its infimum on } A.$$

and

$$|h(d) - g_0^*(d)| < \epsilon$$

for every $d \in D$.

Theorem (Second Birthday Theorem for Domingo 2018)

Let A be a convex bounded subset of a Banach space E. Let us fix a convex weakly compact subset D of E which does not contain the origin, a functional $z_0^* \in E^*$ and $\epsilon > 0$. Then there is a bounded sequence $\{x_n^*\}_{n\geq 1}$ in E^* and $g_0^* \in co_\sigma\{x_n^* : n \geq 1\}$ such that for all $h \in \ell_\infty(A)$ satisfying that for all $a \in A$,

$$\liminf_{n\geq 1} x_n^*(a) \leq h(a) \leq \limsup_{n\geq 1} x_n^*(a),$$

we have that

$$z_0^* + h - g_0^* \text{ does not attain its infimum on } A.$$

and

$$|h(d) - g_0^*(d)| < \epsilon$$

for every $d \in D$.

Theorem (Second Birthday Theorem for Domingo 2018)

Let A be a convex bounded and not relatively compact subset of a Banach space E. Let us fix a convex weakly compact subset D of E, a functional $z_0^* \in E^*$ and $\epsilon > 0$. Then there is a linear form $x_0^* \in B_{p_D}(z_0^*, \epsilon)$, i.e.

$$|x_0^*(d) - z_0^*(d)| < \epsilon$$

for all $d \in D$,
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Reflexivity consequences

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A Banach space E is reflexive if, and only if, the set of norm attaining linear functionals has non empty interior for the Mackey topology $\tau(E^*, E)$ in E^*, (the topology of uniforme convergence on weakly compact convex subsets of E.)

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A Banach space E is not reflexive if, and only if, the set of norm non-attaining linear functionals is dense in E^* for the Mackey topology $\tau(E^*, E)$, (the topology of uniforme convergence on weakly compact convex subsets of E.)

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One-sided plus Mackey's constraints Case

Theorem (Joint work with Freddy Delbaen)

Let A be a convex bounded subset of a Banach space E which is assumed non to be relatively weakly compact. Let us fix an absolutely convex and weakly compact subset D of E^{**} and a functional $z_0^* \in E^*$, with $z_0^*(A) > 0$, and $\epsilon > 0$
Then there are linear forms $z^* \in E^*$ such that

$$z_0^* + z^* \text{ does not attain its infimum on } A,$$

$$|z^*(d)| < \epsilon$$

for every $d \in D$ and

$$(z_0^* + z^*)(A) > 0.$$

Tool: the undetermined function technique again

Theorem (F. Delbaen and J. Orihuela)

Let A be a closed, convex and bounded subset of a Banach space E which is not weakly compact and it doest not contains the origin. Let us fix a convex and weakly compact subset D of E, a functional $z_0^* \in E^*$ with $z_0^*(A) > 0$ and $\epsilon > 0$.
Then there is a bounded sequence $\{x_n^*\}_{n\geq 1}$ in E^* and $g_0^* \in co_\sigma\{x_n^* : n \geq 1\}$ such that for every $\sigma(E^*, E)$-cluster point $h^* \in E^*$ of the sequence $\{x_n^*\}_{n\geq 1}$ we have that

$$z_0^* + h^* - g_0^* \text{ does not attain its infimum on } A,$$

$$|h^*(w) - g_0^*(d)| < \epsilon$$

for every $d \in D$, and

$$(z_0^* + h^* - g_0^*)(A) > 0.$$

Theorem (F. Delbaen and J. Orihuela)

Let C be a proper closed cone in a Banach space E which is assumed non to be $\sigma(E^{**}, E^*)$-closed in E^{**}. Let us fix a convex and weakly compact subset D of E, a functional $z_0^* \in E^*$ with $z_0^*(C) \leq 0$, and $\epsilon > 0$.
Then there is a linear form $z^* \in E^*$ such that

$$x \to \langle z_0^* + z^*, x\rangle \text{ does not attain its supremum on } C,$$

$$\sup_{d \in D} |z^*(d)| < \epsilon,$$

and

$$(z_0^* + z^*)(C) \leq 0.$$

Maximizing $\{x^*(x) - \alpha(x) : x \in E\}$

Theorem (F. Delbaen and J. Orihuela)

Let E be a Banach space, $\alpha : E \to (-\infty, +\infty]$ be a proper, lower semicontinuous function.
If $\partial\alpha(E)$ has non empty interior in E^* for the Mackey topology $\tau(E^*, E)$, then the level sets $\{\alpha \leq c\}$ are weakly compact for all $c \in \mathbb{R}$.
Moreover, if the Fenchel-Legendre conjugate α^* is finite, i.e. $\sup\{x^*(x) - \alpha(x) : x \in E\} < +\infty$ for all $x^* \in E^*$, then $\partial\alpha(E) = E^*$

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