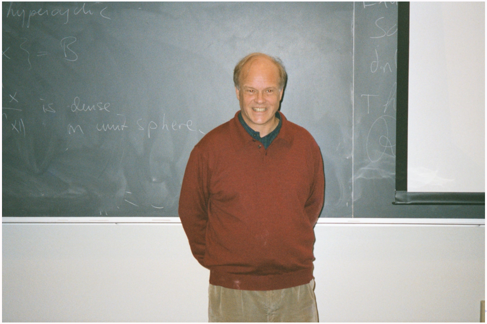
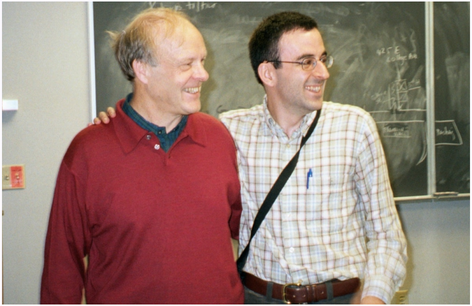
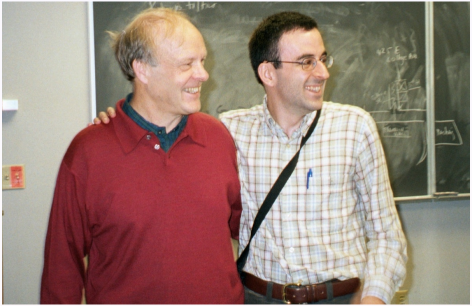
Banach spaces of functions having infinitely many zeros

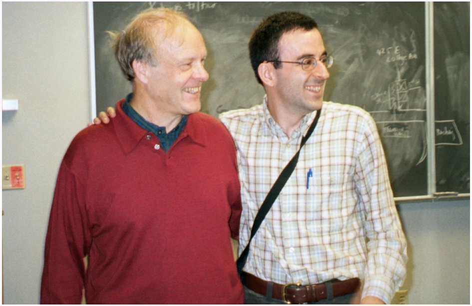
Juan B. Seoane Sepúlveda (UCM, Madrid, Spain)

Per Enflo’s Workshop. *Solving the Invariant Subspace Problem*









Kent, 2001

Valencia, 2023

TRANSACTIONS OF THE
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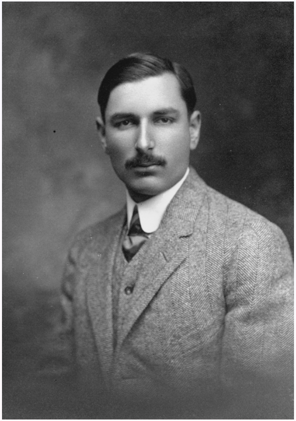
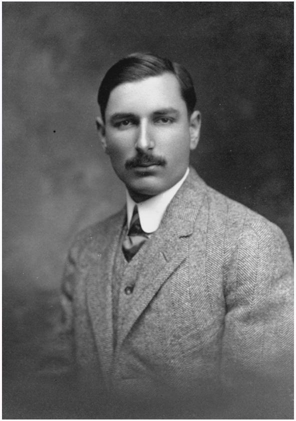


Henry Blumberg (1886–1950)

Sierpiński–Zygmund functions



Henry Blumberg (1886–1950)

Theorem (Blumberg, 1922)

Let $f : \mathbb{R} \to \mathbb{R}$ be an arbitrary function. There exists a dense subset $S \subset \mathbb{R}$ such that the function $f|_S$ is continuous.

A careful reading of the proof of this result shows that the previous set, S, is countable.

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Can we choose the subset S in Blumberg's theorem to be uncountable?

Theorem (Sierpiński–Zygmund, 1923)

There exists a function $f : \mathbb{R} \to \mathbb{R}$ such that, for any set $Z \subset \mathbb{R}$ of cardinality the continuum, the restriction $f|_Z$ is not continuous.

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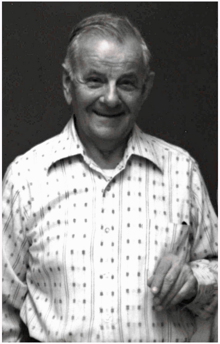

A. Zygmund
(1882–1969)

W. Sierpiński
(1900–1992)

$$\mathcal{SZ}(\mathbb{R}) = \{\, f\colon \mathbb{R} \to \mathbb{R} \;:\; f \text{ is a Sierpiński-Zygmund function} \,\}$$

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Theorem (Gámez, Muñoz, Sánchez, S., 2010)

$\mathcal{SZ}(\mathbb{R})$ is κ-lineable for some cardinal κ with $\mathfrak{c} < \kappa \leq 2^{\mathfrak{c}}$.

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Theorem (Gámez, Muñoz, Sánchez, S., 2010)

$\mathcal{SZ}(\mathbb{R})$ is κ-lineable for some cardinal κ with $\mathfrak{c} < \kappa \leq 2^{\mathfrak{c}}$. Assuming the Generalized Continuum Hypothesis, $\mathcal{SZ}(\mathbb{R})$ is $2^{\mathfrak{c}}$-lineable.

Can the $2^{\mathfrak{c}}$-lineability of $\mathcal{SZ}(\mathbb{R})$ be obtained in ZFC?

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Theorem (Gámez, S., 2014)

The $2^{\mathfrak{c}}$-lineability of $\mathcal{SZ}(\mathbb{R})$ in undecidable.

Non-linear sets of continuous functions

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Let $\widehat{C}[0,1]$ be the subset of $C[0,1]$ of functions admitting one (and only one) absolute maximum.

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In 2005 Gurariy and Quarta proved that $\widehat{\mathcal{C}}(\mathbb{R})$ is 2-lineable.

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Bernal, Cabana, Muñoz, S.,
On the dimension of subspaces of continuous functions attaining their maximum finitely many times.
Trans. Amer. Math. Soc. 373 (2020), 3063–3083.

Annulling functions in $\mathcal{C}[0,1]$

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Definition

A function $f \in \mathcal{C}[0,1]$ *is said to be an* annulling function *if* f *has infinitely many zeros in* $[0,1]$.

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Is the set of annulling functions spaceable in $\mathcal{C}[0,1]$?

Annulling functions and spaceability

Annulling functions and spaceability

Theorem (Enflo, Gurariy, S., 2014)

Let X be any infinite dimensional closed subspace of $\mathcal{C}[0,1]$.
There exists:

- **An infinite dimensional closed subspace Y of X, and**
- **a sequence $\{t_k\}_{k\in\mathbb{N}} \subset [0,1]$ (of pairwise different elements),**

such that $y(t_k) = 0$ for every $k \in \mathbb{N}$ and every $y \in Y$.

Some consequences of the previous result...

Definition

The **oscillation** $O_{[\alpha,\beta]}x$ of $x \in \mathcal{C}[0,1]$ on $[\alpha,\beta]$ is defined as

$$O_{[\alpha,\beta]}x = \sup_{t,s\in[\alpha,\beta]} |x(t) - x(s)|.$$

Let $a > 0$ and $t_0 \in [0,1]$. We say that t_0 is *a*-oscillating for a family of functions $F \subset \mathcal{C}[0,1]$ if for every $d > 0$ there is $x \in F$ such that

$$O_{[0,1]\cap[t_0-d,t_0+d]}x > a.$$

For short, we shall say that t_0 is oscillating if it is $a-$oscillating for some $a > 0$.

In general, the set of all oscillating points ($a-$oscillating points) of a given family $F \subset \mathcal{C}[0,1]$ shall be called the **oscillating spectrum** of F (denoted $\Omega(F)$, or $\Omega_a(F)$, respectively).

Some results on $\Omega(F)$ (Enflo, Gurariy, S., 2014)

- A uniformly bounded set $F \subset \mathcal{C}[0,1]$ is compact if and only if $\Omega(F) = \varnothing$.
- Let X be a subspace of $\mathcal{C}[0,1]$. If $\Omega(X)$ is finite then X is isomorphic to a subspace of c_0.
- For any closed subset $M \subset [0,1]$ there exists a subspace X of $\mathcal{C}[0,1]$ with $M = \Omega(X)$.
- There exists a subspace $X \subset \mathcal{C}[0,1]$ for which $\Omega(X) = (0,1]$.

Q1: Let X be a subspace of $\mathcal{C}[0,1]$. If $\Omega(X)$ is finite then X is isomorphic to a subspace of c_0. What if $\Omega(X)$ is countable?

Q2: Given X, Y subspaces of $\mathcal{C}[0,1]$, how must $\Omega(X)$ and $\Omega(Y)$ be in order to make X and Y non-isomorphic?

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Q4: What are the properties that $\Omega(X)$ should enjoy in order to obtain that X is uncomplemented in $\mathcal{C}[0,1]$?

Just recently, in

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Rui Xie. On the existing set of the oscillating spectrum. **J. Funct. Anal.** 284 (2023), no. 8, P. 109850, 27 pp.

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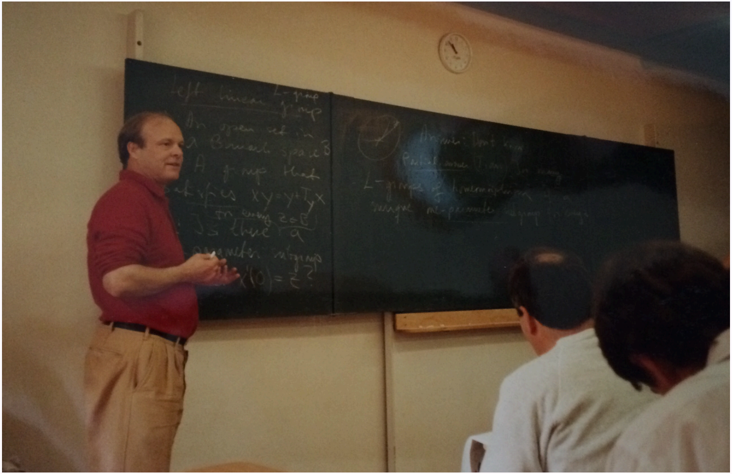
answering **Q3** and providing some info and directions towards **Q4**.



Per in Paseky, 1995.

Thanks for your attention!