Mean ergodic composition operators in spaces of homogeneous polynomials

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V Congreso de Jóvenes Investigadores de la RSME.
Universitat Jaume Primer. January 27-31, 2020

Introduction

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- Let $\varphi : X \to X$ be a holomorphic mapping. We denote by $C_\varphi : \mathcal{P}(^m X) \to H(X)$ the composition operator. It is defined by

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We consider two topologies:

- $\mathcal{P}(^m X)_{\tau_0}$ is a semi-Montel (=every bounded set is rel. compact) lcHs.
- $\mathcal{P}(^m X)_{\|\cdot\|}$ is a Banach space.

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Dynamical properties:

- Power boundedness
- Cesàro boundedness
- Mean ergodicity
- Uniform mean ergodicity

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Let E be a lcHs and $T : E \to E$ and operator. T^n denote the n-th iterate of T. The Cesàro means of T will be defined by

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Let $\mathcal{L}(E)$ denote the space of continuous linear operators from E to E.

Definition

An operator $T : E \to E$ is

- *Power Bounded: $\{T^n\}$ is equicontinuous in $\mathcal{L}(E)$,*
- *Cesàro Bounded: $\{T_{[N]}\}$ is equicontinuous in $\mathcal{L}(E)$,*

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Definition

An operator $T : E \to E$ is

- *Mean Ergodic (ME): $\{T_{[N]}\}$ converges in the topology of pointwise convergence of $\mathcal{L}(E)$ (strong operator topology when E is Banach),*
- *Uniformly Mean Ergodic (UME): $\{T_{[N]}\}$ converges in the topology of bounded convergence of $\mathcal{L}(E)$ (operator norm topology when E is Banach).*

Introduction

Proposition (Bonet, Domański)

Let U be a connected domain of holomorphy in $\mathbb{C}^d$ and let $\varphi : U \to U$ a holomorphic mapping. T.F.A.E.:

- a $C_\varphi : H(U) \to H(U)$ *is power bounded.*
- b $C_\varphi : H(U) \to H(U)$ *is uniformly mean ergodic.*
- c $C_\varphi : H(U) \to H(U)$ *is mean ergodic.*
- d φ *has stable orbits.*

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Fact: $H(U)$ is Fréchet-Montel, but $\mathcal{P}(^m X)_{\tau_0}$ is not barrelled and $\mathcal{P}(^m X)_{\|\cdot\|}$ is not reflexive (Montel) in general.

Preliminary results

Proposition

Let $\varphi : X \to X$ be a holomorphic mapping. The composition operator $C_\varphi : \mathcal{P}(^m X) \to \mathcal{P}(^m X)$ is well defined if and only if φ is linear.

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Remark

The operator $C_\varphi : \mathcal{P}(^m X)_\tau \to \mathcal{P}(^m X)_\tau$ is continuous if $\tau = \tau_0$ or $\| \cdot \|$.

C_φ power bounded in $\mathcal{P}(^mX)_{\tau_0}$

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Lemma

Let $K \subseteq X$ be a compact set. Then

$$\widehat{K}_{\mathcal{P}(^mX)} := \{x \in X : |p(x)| \leq \sup_{y \in K} |p(y)|, \text{ for all } p \in \mathcal{P}(^mX)\}$$

is compact.

Power bounded $\Rightarrow$ UME in $\mathcal{P}(^m X)_{\tau_0}$

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Corollary

Let φ be a continuous linear mapping. If $C_\varphi : \mathcal{P}(^m X)_{\tau_0} \to \mathcal{P}(^m X)_{\tau_0}$ is power bounded, then it is uniformly mean ergodic.

C_φ can be UME and NOT power bounded in $\mathcal{P}(^m X)_{\tau_0}$

Theorem (Bermúdez, Bonilla, Müller, Peris)

There exist mean ergodic and mixing operators on ℓ_p for $1 < p < \infty$.

Let $0 < \alpha < 1/p$, consider $\varphi_\alpha : \ell_p \to \ell_p$ defined by

$$\varphi_\alpha(x_1, x_2, \dots) = (w_1 x_2, w_2 x_3, \dots),$$

where $w_k = \left(\frac{k+1}{k}\right)^\alpha$.

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Example

The composition operator $C_{\varphi_\alpha} : \mathcal{P}(^1\ell_p)_{\tau_0} \to \mathcal{P}(^1\ell_p)_{\tau_0}$ is uniformly mean ergodic, but not power bounded.

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Proposition

Let $\varphi : X \to X$ be a continuous linear map. Then $C_\varphi : \mathcal{P}(^mX)_{\|\cdot\|} \to \mathcal{P}(^mX)_{\|\cdot\|}$ is power bounded if and only if φ is power bounded.

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Let $\varphi : X \to X$ be a continuous linear map such that $C_\varphi : \mathcal{P}(^mX)_{\tau_0} \to \mathcal{P}(^mX)_{\tau_0}$ is power bounded. Then $C_\varphi : \mathcal{P}(^mX)_{\|\cdot\|} \to \mathcal{P}(^mX)_{\|\cdot\|}$ is power bounded.

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The converse is not true in general. One example is C_φ on $\mathcal{P}(^mc_0)$ with the symbol

$$\varphi(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots).$$

C_φ Cesàro bounded in $\mathcal{P}(^mX)_{\|\cdot\|}$

Example

Fix $m \geq 2$ and $0 < \alpha < 1/m$. Then $C_{\varphi_\alpha} : \mathcal{P}(^m\ell_m)_{\|\cdot\|} \to \mathcal{P}(^m\ell_m)_{\|\cdot\|}$ is Cesàro bounded but neither power bounded nor mean ergodic.

Where

$$\varphi_\alpha(x_1, x_2, \ldots) = (w_1 x_2, w_2 x_3, \ldots),$$

with $w_k = \left(\frac{k+1}{k}\right)^\alpha$.

Power Bounded vs Mean Ergodic on $\mathcal{P}(^m X)_{\|\cdot\|}$

Consider the usual backward shift $\sigma : \ell_m \to \ell_m$

Example

$C_\sigma : \mathcal{P}(^m \ell_m)_{\|\cdot\|} \to \mathcal{P}(^m \ell_m)_{\|\cdot\|}$ is power bounded but not mean ergodic.

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$C_\sigma : \mathcal{P}(^m \ell_m)_{\|\cdot\|} \to \mathcal{P}(^m \ell_m)_{\|\cdot\|}$ is power bounded but not mean ergodic.

Fix $1 < p < \infty$ and let $0 < \beta < 1/p'$. Consider $\psi_\beta : \ell_p \to \ell_p$ defined by

$$\psi_\beta(x_1, x_2, \ldots) = (0, w_1 x_1, w_2 x_2, \ldots),$$

where $w_k = \left(\frac{k+1}{k}\right)^\beta$.

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Example

$C_{\psi_\beta} : \mathcal{P}(^1\ell_p)_{\|\cdot\|} \to \mathcal{P}(^1\ell_p)_{\|\cdot\|}$ is mean ergodic, but not power bounded.

Answered questions

M. Maestre

Let $\varphi : X \to X$ be a continuous mapping. The composition operator $C_\varphi : \mathcal{P}(^m X) \to \mathcal{P}(^m X)$ is well defined if and only if φ is linear?

Answered questions

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R. Aron

Let $\varphi : X \to X$ be a holomorphic mapping. The composition operator $C_\varphi : \mathcal{P}(^m X) \to \mathcal{P}(^{m \cdot 2} X)$ is well defined if and only if φ is a 2-homogeneous polynomial?

What can be said if $\varphi : X \to Y$?

Answered questions

Lemma

Let $\varphi : X \to Y$ be a continuous mapping. If there exists $m \in \mathbb{N}$ such that $\gamma^m \circ \varphi$ is holomorphic for every $\gamma \in Y^*$, then φ is holomorphic.

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Lemma

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Proposition

Let $\varphi : X \to Y$ be a continuous mapping, $m \in \mathbb{N}$ and $h \in \mathbb{N}_0$. The composition operator $C_\varphi : \mathcal{P}(^m Y) \to \mathcal{P}(^h X)$ is well defined if and only if

1. *$\varphi \in \mathcal{P}(^k X, Y)$, when $h = k \cdot m$, $k \in \mathbb{N}_0$, or*
2. *$\varphi \equiv 0$ otherwise.*

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