# The twisting, from $C$ to $K$

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**Theorem (Miljutin)**

Given two uncountable metrizable compacta $K$ and $L$, then $C(K) \simeq C(L)$.

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- The same works for

$$c_0(\Gamma) = \{x : \Gamma \to \mathbb{R} : \forall \varepsilon > 0,\ |x(\gamma)| > \varepsilon \text{ for finite } \gamma\}$$

So $c_0(\Gamma) \simeq C(\alpha\Gamma)$.

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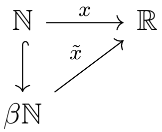

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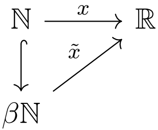


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$\beta\mathbb{N} =$ *Stone-Čech compactification*

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$$Z = Y \oplus X \quad , \quad \|(y,x)\| = \|y\|_Y + \|x\|_X$$

This produces the *trivial twisted sum*.

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***Our question***

What happens to $\operatorname{Ext}(X,Y)$ when $X$ and $Y$ are $C(K)$?

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- $C(K)$ separable $\iff$ $K$ metrizable.
- In particular, if $K$ is metrizable, $\mathrm{Ext}(C(K), c_0) = 0$.

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1. $c_0$ is not complemented in $\ell_\infty$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & c_0 & \longrightarrow & \ell_\infty & \longrightarrow & \ell_\infty/c_0 & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ 0 & \longrightarrow & C(\alpha\mathbb{N}) & \longrightarrow & C(\beta\mathbb{N}) & \longrightarrow & C(\beta\mathbb{N}\setminus\mathbb{N}) & \longrightarrow & 0 \end{array}$$

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2. *Pełczyński's exact sequence:*
There exists an uncomplemented copy of $C[0,1]$ into $C[0,1]$ whose quotient is $c_0$:

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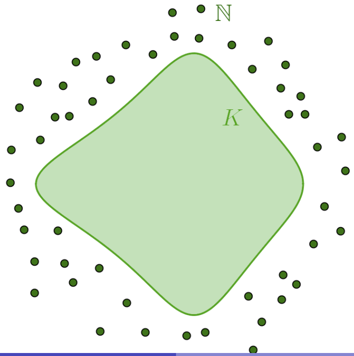

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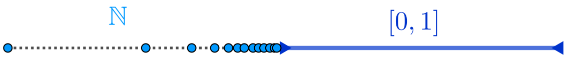


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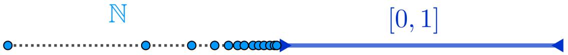


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☞ *When $C(K \cup \mathbb{N})$ is not trivial?*

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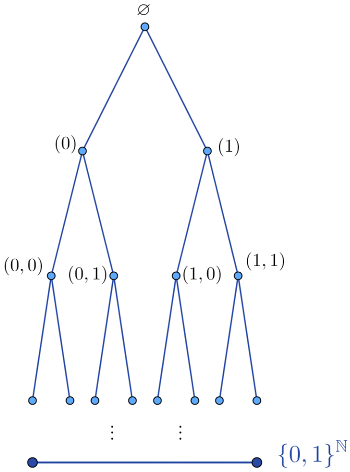

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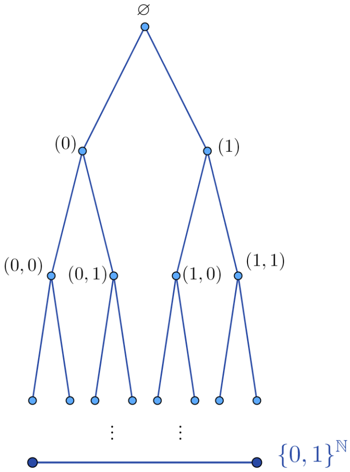


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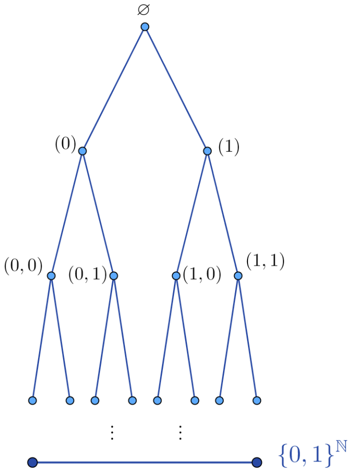


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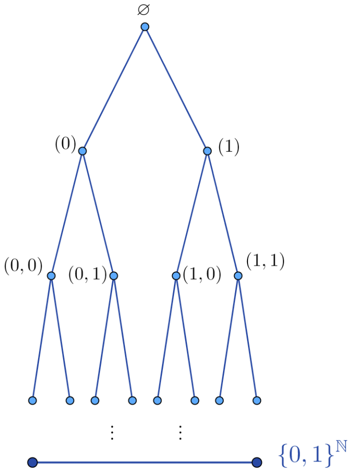


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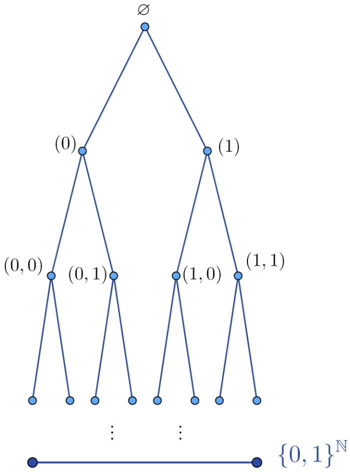


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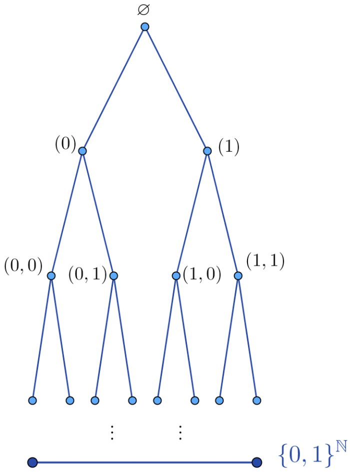


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$K$ is a countable discrete extension of the “set of branches”.

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**Proposition**

- $K_{\mathcal{A}}$ is a countable discrete extension of $\alpha(\mathfrak{c})$.
- $C(K_{\mathcal{A}})$ is a non-trivial twisted sum of $c_0$ and $c_0(\mathfrak{c})$.

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**Theorem 1 (Cabello, Castillo, Marciszewski, Plebanek, S-A.)**

*There are* $2^{\mathfrak{c}}$ *non-isomorphic twisted sums of* $c_0$ *and* $c_0(\mathfrak{c})$*.*
*Moreover, all of them are of the form* $C(K_{\mathcal{A}})$*.*

And this is because...

**Lemma.** A family $\mathcal{K}$ of non-homeomorphic separable compact spaces satisfying:

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☞ Apply this to $\{K_{\mathcal{A}}\}$ for $|\mathcal{A}| = \mathfrak{c}$.

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— *But what about "normal" $C(K)$'s?*

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**Theorem 4 (Marciszewski, Plebanek)**

[MA + ¬ CH] *If $|\mathcal{A}| = \mathfrak{c}$, then Ext$(C(K_{\mathcal{A}}), c_0) = 0$.*

THANK YOU
FOR YOUR ATTENTION