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Smooth renormings on dense subspaces of Banach spaces

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References and Acknowledgements

[DHR] S. Dantas, P. Hájek, and T. Russo
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MINISTERSTVO ŠKOLSTVÍ, MLÁDEŽE A TĚLOVÝCHOVY

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A couple of references:

[DGZ] Deville–Godefroy–Zizler, *Smoothness and Renorming*...;

[HJ] Hájek–Johanis, *Smooth Analysis in Banach spaces*.

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- **(Rather bold?) Problem.** Given a Banach space X, is there a dense subspace of X that admits a C^k-smooth norm?
    - Yes, if X is separable (Hájek, 1995).

Let ℓ_∞^F be the dense subspace of ℓ_∞ comprising sequences that assume finitely many values, *i.e.*,

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(i) Every normed space with a countable Hamel basis admits an analytic norm (and analytic norms are dense);

(ii) The space of finitely supported vectors in $\ell_1(\mathfrak{c})$ admits an analytic norm.

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Let $\ell_\infty^{F,c}(\Gamma)$ be the dense subspace of $\ell_\infty^c(\Gamma)$ comprising functions with countable support and that assume finitely many values, *i.e.*,

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- Compare $c_0(\omega_1)$ and $\ell_1(\omega_1)$.

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- Let X be a Banach space such that every dense subspace contains a further dense subspace with a C^k-smooth norm.
  What can we say about X?

But not today

Thank you for your attention!