Strong norm attainment and applications

Abraham Rueda Zoca
(joint with B. Cascales, R. Chiclana, L. García-Lirola and M. Martín)

V Congreso de Jóvenes Investigadores de la RESME

Universidad de Granada
Departamento de Análisis Matemático

UNIVERSIDAD DE GRANADA

Support

My research is supported by Vicerrectorado de Investigación y Transferencia de la Universidad de Granada in the program “Contratos puente”, by MICINN (Spain) Grant PGC2018-093794-B-I00 (MCIU, AEI, FEDER, UE), by Junta de Andalucía Grant A-FQM-484-UGR18 and by Junta de Andalucía Grant FQM-0185.

Collaborators

Srongly norm-attaining Lipschitz maps

Given a **complete** metric space (M, d) with a distinguished point $0 \in M$ and a Banach space Y, we consider the space

$$\text{Lip}_0(M, Y) := \{f : M \to Y : f \text{ is Lipschitz}, f(0) = 0\}$$

which is a Banach space when equipped with the norm

$$\|f\|_L := \sup\left\{\frac{\|f(x) - f(y)\|}{d(x, y)} : x \neq y\right\}.$$

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We say that f **strongly attains its norm** if

$$\|f\|_L = \frac{\|f(x) - f(y)\|}{d(x, y)}$$

for some $x, y \in M$. We denote $\mathrm{SNA}(M, Y)$ the set of such maps.

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Problem

When $\overline{\mathrm{SNA}(M, Y)} = \mathrm{Lip}_0(M, Y)$?

Negative results

Theorem (V. Kadets, M. Martín, M. Soloviova, 2016)

If M is geodesic (in particular, $M = [0,1]$) then $\overline{\mathrm{SNA}(M,\mathbb{R})} \neq \mathrm{Lip}_0(M,\mathbb{R})$.

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- *M is a length space (i.e. $d(x, y)$ is the infimum of the length of curves joining x and y, for every x, y).*

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Theorem (R. Chiclana, L. García-Lirola, M. Martín and A.R.Z., 2019)

$\overline{\mathrm{SNA}(\mathbb{T},\mathbb{R})} \neq \mathrm{Lip}_0(\mathbb{T},\mathbb{R})$, *where $\mathbb{T}$ denotes the unit circle in $\mathbb{R}^2$.*

Lispchitz-free spaces

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$$\mathcal{F}(M)^* = \mathrm{Lip}_0(M).$$

Linearisation of Lipschitz mappings

Given a metric space M, a Banach space Y and a Lispchitz map $f : M \longrightarrow Y$ such that $f(0) = 0$, there exists a bounded operator $\hat{f} : \mathcal{F}(M) \longrightarrow Y$ defined by

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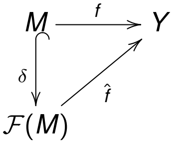
This operator $\hat{f}$ satisfies that $\|\hat{f}\| = \|f\|_L$ and that the following diagram is conmutative



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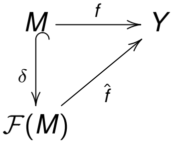

From here it follows that the mapping

$$\begin{array}{ccc} \mathrm{Lip}_0(M, Y) & \longrightarrow & L(\mathcal{F}(M), Y), \\ f & \longmapsto & \hat{f} \end{array}$$

is an onto linear isometry, so $\mathrm{Lip}_0(M, Y) = L(\mathcal{F}(M), Y)$.

Positive results I

Note that, if f strongly attains its norm at $x, y \in M$, then

$$\left\| \hat{f}\left(\frac{\delta(x) - \delta(y)}{d(x,y)} \right) \right\| = \|f\|_L$$

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Theorem (Godefroy, 2015)

Assume M is a compact metric space and $\mathrm{lip}_0(M)^* = \mathcal{F}(M)$. Then $\overline{\mathrm{SNA}(M, Y)} = \mathrm{NA}(\mathcal{F}(M), Y)$ for all Y. Moreover, if Y is finite-dimensional, then $\overline{\mathrm{SNA}(M, Y)} = \mathrm{Lip}_0(M, Y)$.

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3. *denting point* if for every $\varepsilon > 0$ there exists a slice $S = S(B_X, f, \alpha) := \{y \in B_X : f(y) > 1 - \alpha\}$ such that $x \in S$ and that $\operatorname{diam}(S) < \varepsilon$.

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4. *strongly exposed point* if there exists a functional $f \in S_{X^*}$ such that $f(x) = 1$ and $\inf_{\alpha>0} \text{diam}\,(S(B_X, f, \alpha)) = 0$.

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It is known that

strongly exposed $\Rightarrow$ denting $\Rightarrow$ preserved extreme $\Rightarrow$ extreme

Positive results II

$\overline{\text{SNA}(M, Y)} = \text{Lip}_0(M, Y)$ regardless Y whenever:

1. $\mathcal{F}(M)$ has the RNP (L. García-Lirola, C. Petitjean, A. Procházka, A. R. Z., 2018).

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2. The unit ball of $\mathcal{F}(M)$ contains a norming uniformly strongly exposed set (in particular if $\mathcal{F}(M)$ has property α) (B. Cascales, R. Chiclana, L. García-Lirola, M. Martín and A. R. Z., 2019).

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In all the above cases a Banach space property (involving an abundance of strongly exposed points of the unit ball) of $\mathcal{F}(M)$ implies the (Lipschitz) property of density of the set of strongly norm-attaining Lipschitz maps.

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Concerning Question 1 we have the following result:

Theorem (R. Chiclana, L. García-Lirola, M. Martín and A. R. Z., 2019)

There exists a compact metric space M such that:

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There exists a compact metric space M such that:

1. $\overline{\mathrm{SNA}(M, Y)} = \mathrm{Lip}_0(M, Y)$ *for every* Y.
2. $\mathcal{F}(M)$ *contains an isometric copy of* $L_1([0, 1])$ *(in particular* $\mathcal{F}(M)$ *fails the RNP).*

Strongly exposed points I

Theorem (R. Chiclana-L. García-Lirola,M. Martín, A. R. Z., 2019)

Assume that M is compact and than $\overline{\mathrm{SNA}(M,\mathbb{R})}$ is dense in $\mathrm{Lip}_0(M,\mathbb{R})$. Then $\mathrm{SNA}(M,\mathbb{R})$ contains an ***open*** *dense subset.*

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Proof. Let

$$A=\{f\in \mathrm{Lip}_0(M,\mathbb{R}) : \sup_{d(x,y)<\varepsilon}\frac{f(x)-f(y)}{d(x,y)}<\|f\|_L \text{for some } \varepsilon>0\}$$

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By Aliaga-Pernecká, we may assume that $\frac{\delta(x)-\delta(y)}{d(x,y)} \in \text{ext}(B_{\mathcal{F}(M)})$.

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Now, by Aliaga-Guirao and García-Lirola, Petitjean, Procházka and R.Z.

$$\text{ext}(B_{\mathcal{F}(M)}) = \text{ext}(B_{\mathcal{F}(M)^{**}}) \cap \mathcal{F}(M) = \text{dent}(B_{\mathcal{F}(M)})$$

Strongly exposed points II

Therefore, there is $g \in S_{\mathrm{Lip}_0(M)}$ and $\beta > 0$ such that $\frac{g(x)-g(y)}{d(x,y)} > 1-\beta$ and $\operatorname{diam}\{\mu \in B_{\mathcal{F}(M)} : g(\mu) > 1-\beta\} < \varepsilon$.

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Take $h = f + \varepsilon g$. Then $\|f - h\|_L = \varepsilon$. We claim that $h \in A$.

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So, $g\left(\frac{\delta(u)-\delta(v)}{d(u,v)}\right) > 1-\beta$ and thus $\|\frac{\delta(u)-\delta(v)}{d(u,v)} - \frac{\delta(x)-\delta(y)}{d(x,y)}\| < \varepsilon$.

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A result of García-Lirola thesis implies that $d(u,v) \geq (1-2\varepsilon)d(x,y)$, that is, h does not approximate its norm at arbitrarily close points.

Strongly exposed points III

$$A = \{f \in \text{Lip}_0(M, \mathbb{R}) : \sup_{d(x,y)<\varepsilon} \frac{f(x) - f(y)}{d(x,y)} < \|f\|_L \text{for some } \varepsilon > 0\}$$

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Corollary

If M is compact and $\overline{\mathrm{SNA}(M, \mathbb{R})}$ is dense in $\mathrm{Lip}_0(M, R)$, then

$$B_{\mathcal{F}(M)} = \overline{co}(\mathrm{strexp}(B_{\mathcal{F}(M)})).$$

Strongly exposed points III

$$A = \{f \in \mathrm{Lip}_0(M, \mathbb{R}) : \sup_{d(x,y)<\varepsilon} \frac{f(x) - f(y)}{d(x, y)} < \|f\|_L \text{for some } \varepsilon > 0\}$$

By an adaptation of a result of Y. Ivakhno, V. Kadets and D. Werner (2007) we get that every element $f \in A$ strongly attain its norm at a strongly exposed point. Hence:

Corollary

If M is compact and $\overline{\mathrm{SNA}(M, \mathbb{R})}$ is dense in $\mathrm{Lip}_0(M, R)$, then

$$B_{\mathcal{F}(M)} = \overline{co}(\mathrm{strexp}(B_{\mathcal{F}(M)})).$$

The converse is not true. Every element of the form $\frac{\delta_x - \delta_y}{d(x,y)}$ is a strongly exposed point in $\mathcal{F}(\mathbb{T})$ and, as we have seen, $\mathrm{SNA}(\mathbb{T}, \mathbb{R})$ is not dense in $\mathrm{Lip}_0(\mathbb{T}, \mathbb{R})$.

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Thank you


THANK YOU FOR YOUR ATTENTION
PLEASE CLAP AND DO NOT MAKE TOUGH QUESTIONS