ESTIMATES OF LACUNARY NORMS OF PRODUCTS OF POLYNOMIALS

Daniel L. Rodríguez-Vidanes
UCM. Madrid, España
Joint work with G. Araújo, P. H. Enflo, G. A. Muñoz-Fernández and J. B. Seoane-Sepúlveda

V Congreso Jóvenes Investigadores de la RSME
30 January 2019

Background

Conjecture (The invariant subspace problem)

Let X be a complex Banach space of dimension > 1. If $T : X \to X$ is a bounded linear operator, then X has a closed non-trivial T-invariant subspace.

Background

Conjecture (The invariant subspace problem)

Let X be a complex Banach space of dimension > 1. If $T : X \to X$ is a bounded linear operator, then X has a closed non-trivial T-invariant subspace.

Theorem (Enflo - 1987)

There exists a Banach space X and a bounded linear operator $T : X \to X$ such that X does not have non-trivial T-invariant subspaces.

Estimates on norms of polynomials

Remark (The technique)

One of the many ideas that Enflo used was the "concentration of low degree polynomials" by using a series of estimates on the products of polynomials. For instance, if P and Q are polynomials of degrees n_1 and n_2, respectively, then there exists $C(n_1, n_2)$ such that

$$|PQ| \geq C(n_1, n_2)|P||Q|$$

where $|\cdot|$ denotes the sum of the absolute values of the coefficients of a polynomial and $C(n_1, n_2)$ is independent on the number of variables.

Estimates on norms of polynomials

Definition (Concentration of a polynomial at low degrees)

In one variable, let $q = \sum_{j \geq 0} a_j z^j$ be a polynomial with complex coefficients and $0 \leq d \leq 1$. We say that q has concentration d, at degree at most k, if

$$\sum_{j=0}^{k} |a_j| \geq d \sum_{j \geq 0} |a_j|.$$

Estimates on norms of polynomials

Definition (Concentration of a polynomial at low degrees)

In one variable, let $q = \sum_{j\geq 0} a_j z^j$ be a polynomial with complex coefficients and $0 \leq d \leq 1$. We say that q has concentration d, at degree at most k, if

$$\sum_{j=0}^{k} |a_j| \geq d \sum_{j\geq 0} |a_j|.$$

Remark

Extension of the definition of concentration of polynomials at low degrees to other norms defined on the vector spaces of homogeneous polynomials.

Standard polynomial norms

Notation

Let $q = \sum_{j \geq 0} a_j z^j$. We consider the following norms:

(i) $\|q\|_1 = \sum_{j \geq 0} |a_j|$,

(ii) $\|q\|_{\sup} = \sup_{|z|=1} |q(z)|$,

(iii) $\|q\|_2 = \left(\sum_{j \geq 0} |a_j|^2\right)^{\frac{1}{2}}$,

(iv) $\|q\|_{L_1} = \frac{1}{2\pi} \int_{|z|=1} |q(z)| d\theta$.

Notice that $\|q\|_1 \geq \|q\|_2 \geq \|q\|_{L_1} \geq \|q\|_{\sup}$. Also let us denote for $n \in \mathbb{N}$,

$$q|_n = \sum_{j=0}^{n} a_j z^j.$$

Lacunary norms

Definition (Lacunary sets)

(a) We say that a singleton containing a non-negative integer is a 0-lacunary subset of the integers.

(b) Inductively, given an integer k, we define a k-lacunary subset of the integers as follows:
E is a k-lacunary subset of the integers if the intersection between E and a translate of E is never bigger than a $(k-1)$-lacunary subset of E.

We denote the set of all k-lacunary subset of the integers by Ω_k.

Lacunary norms

Example

(i) The infinite set $\{2^k : k \in \mathbb{N}\}$ is 1-lacunary and the infinite set $\{2^k : k \in \mathbb{N}\} \cup \{3^k : k \in \mathbb{N}\}$ is not 1-lacunary.

(ii) Given positive integers n and k, then

$$A_k^a := \{a + in\}_{i=0}^k \in \Omega_k \setminus \Omega_{k-1}$$

for all integers $a \geq 0$.

Lacunary norms

Definition (Lacunary norm)

Let $q = \sum_{j\geq 0} a_j z^j$, we define the k-lacunary norm as

$$\|q\|_{k-\mathrm{lac}} = \sup_{E\in\Omega_k} \{\|q|_E\|_1\}.$$

Notice that $\|q\|_1 \geq \|q\|_{k-\mathrm{lac}}$ for every $k \in \mathbb{N}$.

Auxiliary result

Proposition (Beauzamy and Enflo - 1985)

Given n, C, K, there is $\alpha = \alpha(n, C, K) > 0$ such that for all h and q satisfying

$$\|h\|_{\text{sup}} \leq K\|h\|_2$$
$$\|q\|_2 \leq C\|q|_n\|_2,$$

we have

$$\|hq\|_{L_1} \geq \alpha(n, C, K)\,\|h\|_2\|q\|_2.$$

Main result

Theorem (Araújo, Enflo, Muñoz, Rodríguez and Seoane - 2019/2020)

Given n, C, K, i, and $Q > 1$, there is a $\beta = \beta(n, C, K, Q, i) > 0$ such that for all polynomials h and q satisfying

$$\|h\|_{i-\mathrm{lac}} \leq Q|h_0|,$$
$$\|h\|_1 \leq K\|h\|_{i-\mathrm{lac}},$$
$$\|q\|_1 \leq C\|q|_n\|_1,$$

where $h_0 \neq 0$ is the independent term of h, we have

$$\|hq\|_{i-\mathrm{lac}} \geq \beta(n, C, K, Q, i)\, \|h\|_1 \|q\|_1.$$

Main result

Do we have the following more general result?

Question

Given $n, C, K, i,$, there is a $\beta = \beta(n, C, K, i) > 0$ such that for all polynomials h and q satisfying

$$\|h\|_1 \leq K\|h\|_{i-\mathrm{lac}},$$
$$\|q\|_1 \leq C\|q|_n\|_1,$$

we have

$$\|hq\|_{i-\mathrm{lac}} \geq \beta(n, C, K, i)\,\|h\|_1\|q\|_1.$$

Extension of polynomial norms to Banach spaces

Theorem (Borwein and Erdélyi - 1995)

Let $m, n \in \mathbb{N}$ be such that $m < n$ and let $p \in \mathcal{P}_m^c$ and $q \in \mathcal{P}_{n-m}^c$, then we have

$$\|p\|\|q\| \leq \frac{1}{2} C_{n,m} C_{n,n-m} \|pq\|,$$

where $C_{n,k} = 2^k \prod_{j=1}^{k} \left(1 + \cos \frac{(2j-1)\pi}{2n}\right)$ for $1 \leq k \leq n$. Moreover, the above inequality is sharp in the case where p vanishes at the m roots of T_n closest to -1 and q vanishes at the remaining roots of T_n.

Extension of polynomial norms to Banach spaces

Theorem (Araújo, Enflo, Muñoz, Rodríguez and Seoane - 2019/2020)

Let E be a Banach space over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). Let $m, n \in \mathbb{N}$ be such that $m < n$ and let $P \in \mathcal{P}_m(E)$ and $Q \in \mathcal{P}_{n-m}(E)$. Then we have

$$\|P\|\|Q\| \leq \frac{1}{2} C_{n,m} C_{n,n-m} \|PQ\|,$$

where $C_{n,k}$ with $1 \leq k \leq n$ is as in the theorem above.

Bibliography

[1] G. Araújo, P. H. Enflo, G. A. Muñoz-Fernández, D. L. Rodríguez-Vidanes, and J. B. Seoane-Sepúlveda, *Quantitative and qualitative estimates on the norm of products of polynomials on the norm*, Israel Journal of Mathematics **in press**.

[2] B. Beauzamy and P. Enflo, *Estimations de produits de polynômes*, J. Number Theory **21** (1985), no. 3, 390–412 (French).

[3] P. Borwein and T. Erdélyi, *Polynomials and polynomial inequalities*, Graduate Texts in Mathematics, vol. 161, Springer-Verlag, New York, 1995.

[4] P. Enflo, *On the invariant subspace problem for Banach spaces*, Acta Math. **158** (1987), no. 3-4, 213–313.

THANK YOU!