Numerical index with respect to an operator

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Numerical range for Hilbert spaces (Toeplitz, 1918)

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Obviously one has $v(T) \leqslant \|T\|$

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Set of values

$$\{n(X) \colon X \text{ complex Banach space }\} = [e^{-1}, 1]$$

$$\{n(X) \colon X \text{ real Banach space }\} = [0, 1]$$

Some known results

- H Hilbert space, $n(H) = 0$ in real case and $n(H) = 1/2$ in complex case.

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- $n(X)=1$ iff $\max\limits_{|\theta|=1}\|\mathrm{Id}+\theta\,T\|=1+\|T\|\ \forall T\in\mathcal{L}(X)$ (Duncan et al., 1970)

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- $n(X) = 1$ iff $\max\limits_{|\theta|=1} \|\mathrm{Id} + \theta\, T\| = 1 + \|T\|\ \forall T \in \mathcal{L}(X)$ (Duncan et al., 1970)
- Let $\{X_\lambda \colon \lambda \in \Lambda\}$ be an arbitrary family of Banach spaces. Then

$$n\Big(\big[\oplus_{\lambda\in\Lambda} X_\lambda\big]_{c_0}\Big) = n\Big(\big[\oplus_{\lambda\in\Lambda} X_\lambda\big]_{\ell_1}\Big) = n\Big(\big[\oplus_{\lambda\in\Lambda} X_\lambda\big]_{\ell_\infty}\Big) = \inf_{\lambda\in\Lambda} n(X_\lambda)$$

$$n\Big(\big[\oplus_{\lambda\in\Lambda} X_\lambda\big]_{\ell_p}\Big) \leqslant \inf_{\lambda\in\Lambda} n(X_\lambda)$$

(Martín-Payá, 2000)

Some known results

- Let X be a Banach space, K compact Hausdorff and μ positive measure. Then

$$n\big(C(K,X)\big) = n\big(L_1(\mu,X)\big) = n(X) \qquad \text{(Martín-Payá, 2000)}$$

$$n\big(L_\infty(\mu,X)\big) = n(X) \qquad \text{(Martín-Villena, 2003)}$$

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- $n(L_p(\mu)) = n(\ell_p)$ if $\dim L_p(\mu) = \infty$ (EdDari-Khamsi, 2006)
- $n(L_p(\mu)) > 0$ for $p \neq 2$ (Martín-Merí-Popov, 2011)
- $n(X^*) \leqslant n(X)$
  and the inequality can be strict (Boyko-Kadets-Martín-Werner, 2007)

Extending the concept of numerical range

Intrinsic numerical range

(Bonsall-Duncan, 1971)

Let X be a Banach space. Then for every $T \in \mathcal{L}(X)$

$$\overline{\operatorname{conv}}\, V(T) = \{\Phi(T) \colon \Phi \in \mathcal{L}(X)^*,\ \|\Phi\| = \Phi(\mathrm{Id}) = 1\}.$$

Consequently, $v(T) = \max\{|\Phi(T)| \colon \Phi \in \mathcal{L}(X)^*,\ \|\Phi\| = \Phi(\mathrm{Id}) = 1\}$.

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Intrinsic numerical range with respect to G

X, Y Banach spaces, $G \in \mathcal{L}(X,Y)$ with $\|G\| = 1$, $T \in \mathcal{L}(X,Y)$

$$\widetilde{V}_G(T) = \{\Phi(T) \colon \Phi \in \mathcal{L}(X,Y)^*,\ \|\Phi\| = \Phi(G) = 1\}$$

Spatial numerical range

Spatial numerical range, Bauer–Lumer

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For $G = \mathrm{Id}$, by Bishop–Phelps–Bollobás theorem

$$V_{\mathrm{Id}}(T) = \overline{V(T)} \qquad \text{for every } T \in \mathcal{L}(X)$$

Relationship

Two possible numerical ranges

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Numerical radius with respect to G

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$$v_G(T) = \sup\{|\lambda| \colon \lambda \in V_G(T)\} = \sup\{|\lambda| \colon \lambda \in \widetilde{V}_G(T)\}$$

Numerical index with respect to an operator

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X, Y Banach spaces, $G \in \mathcal{L}(X,Y)$ with $\|G\| = 1$

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We recover the classic numerical index

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Characterization

For $k \in [0,1]$, TFAE:

- $n_G(X,Y) \geqslant k$,
- $\inf_{\delta>0} \sup \{|y^*(Tx)| \colon y^* \in S_{Y^*}, x \in S_X, \operatorname{Re} y^*(Gx) > 1-\delta\} \geqslant k\|T\| \; \forall T \in \mathcal{L}(X,Y)$,
- $\max_{|\theta|=1} \|G + \theta T\| \geqslant 1 + k\|T\| \quad \forall T \in \mathcal{L}(X,Y)$.

Examples

Spear operators (Ardalani, 2014; Kadets, Martín, Merí, Pérez, 2018)

X, Y Banach spaces, $G \in \mathcal{L}(X,Y)$.

G spear operator $\iff n_G(X,Y) = 1 \iff \max_{|\theta|=1} \|G + \theta T\| = 1 + \|T\| \ \forall T \in \mathcal{L}(X,Y)$.

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Examples of spear operators

- Fourier transform ($\mathcal{F} \colon L_1(\mathbb{R}) \longrightarrow C_0(\mathbb{R})$),
- Inclusion $A(\mathbb{D}) \hookrightarrow C(\mathbb{T})$.
- Identity operator on $C(K)$, $L_1(\mu)$...

Set of values

X, Y Banach spaces

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Hilbert spaces

H Hilbert space with $\dim(H) \geqslant 2$, X, Y Banach spaces:

- Real case: $\mathcal{N}(\mathcal{L}(X,H)) = \mathcal{N}(\mathcal{L}(H,Y)) = \{0\}$. In particular, $\mathcal{N}(\mathcal{L}(H)) = \{0\}$.
- Complex case: $\mathcal{N}(\mathcal{L}(X,H)) \subset [0,1/2]$ and $\mathcal{N}(\mathcal{L}(Y,H)) \subset [0,1/2]$.

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H_1, H_2 complex Hilbert spaces with dimension greater than one:

- $\mathcal{N}(\mathcal{L}(H_1,H_2)) = \{0,1/2\}$ if H_1 and H_2 are isometrically isomorphic.
- $\mathcal{N}(\mathcal{L}(H_1,H_2)) = \{0\}$ otherwise.

Set of values

ℓ_p-spaces

For $1 < p < \infty$, X, Y real Banach spaces, $M_p = \sup_{t\in[0,1]} \frac{|t^{p-1}-t|}{1+t^p}$,

$$\mathcal{N}(\mathcal{L}(X,\ell_p)) \subset [0, M_p] \quad \text{and} \quad \mathcal{N}(\mathcal{L}(\ell_p, Y)) \subset [0, M_p].$$

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$$\mathcal{N}(\mathcal{L}(C[0,1], C[0,1])) = \{0,1\} \quad \text{(real case)}$$

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L_∞-spaces

μ_1, μ_2 σ-finite measures. If at least one of the spaces $L_\infty(\mu_i)$, $i = 1, 2$, has dimension at least two, $\mathcal{N}(\mathcal{L}(L_\infty(\mu_1), L_\infty(\mu_2)) = \{0,1\}$ (real and complex case).

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L_1-spaces

μ_1, μ_2 σ-finite measures, $\mathcal{N}(\mathcal{L}(L_1(\mu_1), L_1(\mu_2)) \subset \{0,1\}$ (real and complex case).

★ $\mathcal{N}(\mathcal{L}(\ell_1, L_1[0,1])) = \{0\}$.

Sums of Banach spaces

Proposition

Let $\{X_\lambda \colon \lambda \in \Lambda\}$, $\{Y_\lambda \colon \lambda \in \Lambda\}$ be two families of Banach spaces and let $G_\lambda \in \mathcal{L}(X_\lambda, Y_\lambda)$ with $\|G_\lambda\| = 1$ for every $\lambda \in \Lambda$. Let E be one of the Banach spaces c_0, ℓ_∞ or ℓ_1, let $X = \left[\oplus_{\lambda\in\Lambda} X_\lambda\right]_E$ and $Y = \left[\oplus_{\lambda\in\Lambda} Y_\lambda\right]_E$ and define the operator $G \colon X \longrightarrow Y$ by

$$G\left[(x_\lambda)_{\lambda\in\Lambda}\right] = (G_\lambda x_\lambda)_{\lambda\in\Lambda}$$

for every $(x_\lambda)_{\lambda\in\Lambda} \in \left[\oplus_{\lambda\in\Lambda} X_\lambda\right]_E$. Then

$$n_G(X, Y) = \inf_\lambda n_{G_\lambda}(X_\lambda, Y_\lambda).$$

Sums of Banach spaces

Proposition

Let $\{X_\lambda \colon \lambda \in \Lambda\}$, $\{Y_\lambda \colon \lambda \in \Lambda\}$ be two families of Banach spaces and let $G_\lambda \in \mathcal{L}(X_\lambda, Y_\lambda)$ with $\|G_\lambda\| = 1$ for every $\lambda \in \Lambda$. Let E be one of the Banach spaces c_0, ℓ_∞ or ℓ_1, let $X = [\oplus_{\lambda\in\Lambda} X_\lambda]_E$ and $Y = [\oplus_{\lambda\in\Lambda} Y_\lambda]_E$ and define the operator $G \colon X \longrightarrow Y$ by

$$G\left[(x_\lambda)_{\lambda\in\Lambda}\right] = (G_\lambda x_\lambda)_{\lambda\in\Lambda}$$

for every $(x_\lambda)_{\lambda\in\Lambda} \in [\oplus_{\lambda\in\Lambda} X_\lambda]_E$. Then

$$n_G(X,Y) = \inf_\lambda n_{G_\lambda}(X_\lambda, Y_\lambda).$$

Moreover, for $1 < p < \infty$

$$n_G\left([\oplus_{\lambda\in\Lambda} X_\lambda]_{\ell_p}, [\oplus_{\lambda\in\Lambda} Y_\lambda]_{\ell_p}\right) \leqslant \inf_\lambda n_{G_\lambda}(X_\lambda, Y_\lambda).$$

Vector-valued function spaces

Theorem

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- K compact, consider $\widetilde{G}\colon C(K,X) \longrightarrow C(K,Y)$, $\widetilde{G}(f) = G \circ f$, then:

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Moreover, for vector-valued L_p-spaces
$$n_{\widetilde{G}}\big(L_p(\mu,X), L_p(\mu,Y)\big) \leqslant n_G(X,Y)$$
for $1 < p < \infty$, with $\widetilde{G}$ analogously defined.

Adjoint operators

Numerical index with respect to adjoint operators

X, Y Banach spaces, $G \in \mathcal{L}(X, Y)$ with $\|G\| = 1$,

$$n_{G^*}(Y^*, X^*) \leqslant n_G(X, Y)$$

and this inequality can be strict.

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Proposition

X, Y Banach spaces, $G \in \mathcal{L}(X,Y)$ rank-one operator of norm 1. Then $n_{G^*}(Y^*, X^*) = n_G(X,Y)$ and so, same happens to all the successive adjoints of G.

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