Δ- and Daugavet-points in Banach spaces

Katriin Pirk

University of Tartu, Estonia

A joint work with R. Haller and T. Veeorg

January 28th 2020

Castellón

Supporters

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Delta building

 - points

References

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Notations

In the following let X be a real infinite dimensional Banach space. We use standard notation. Let B_X be closed unit ball and S_X the unit sphere and X^* the dual of X.

We consider a slice of B_X to be a set

$$S(B_X, x^*, \alpha) = \{x \in B_X : x^*(x) > 1 - \alpha\},$$

where $x^* \in S_{X^*}$ and $\alpha > 0$.

For a $x \in S_X$ and $\varepsilon > 0$ we denote by $\Delta_\varepsilon(x)$ the set

$$\Delta_\varepsilon(x) = \{y \in B_X : \|x - y\| \geq 2 - \varepsilon\}.$$

Daugavet property

Proposition 1 (see Werner, 2001)

The following assertions about a Banach space X are equivalent:

(a) X *has the* *Daugavet property*, *i.e.,*

$$\|Id - T\| = 1 + \|T\|$$

for every rank-1 (and norm-1) operator $T: X \to X$;

(b) *for every slice* S *of* B_X, *every* $x \in S_X$ *and every* $\varepsilon > 0$ *there exists an* $y \in S$ *such that* $\|x - y\| \geq 2 - \varepsilon$;

(c) $B_X = \overline{\operatorname{conv}}\, \Delta_\varepsilon(x)$ *for all* $x \in S_X$ *and* $\varepsilon > 0$, *where*

$$\Delta_\varepsilon(x) = \{y \in B_X : \|x - y\| \geq 2 - \varepsilon\}.$$

DLD2P

The Daugavet property implies that every rank-1 projection $P\colon X \to X$ satisfies $\|I - P\| \geq 2$.

Proposition 2 (Ivakhno, Kadets, 2004, and Werner, 2001)

The following assertions about a Banach space X are equivalent:

(a) *X has the diametral local diameter-2 property (DLD2P), i.e.,*

$$\|Id - P\| \geq 2$$

for every rank-1 projection $P\colon X \to X$;

(b) *for every slice S of B_X, every $x \in S \cap S_X$ and every $\varepsilon > 0$ there exists an $y \in S$ such that $\|x - y\| \geq 2 - \varepsilon$;*

(c) *$x \in \overline{\mathrm{conv}}\, \Delta_\varepsilon(x)$ for all $x \in S_X$ and $\varepsilon > 0$, where*

$$\Delta_\varepsilon(x) = \{y \in B_X \colon \|x - y\| \geq 2 - \varepsilon\}.$$

Daugavet points and Δ-points

Motivated by the previous characterizations we introduce the following definitions:

Definition 1

We say that $x \in S_X$ is a *Daugavet point* if $B_X = \overline{\text{conv}}\, \Delta_\varepsilon(x)$ for every $\varepsilon > 0$.

Definition 2

We say that $x \in S_X$ is a *Δ-point* if $x \in \overline{\text{conv}}\, \Delta_\varepsilon(x)$ for every $\varepsilon > 0$.

Recall that $\Delta_\varepsilon(x) = \{y \in B_X : \|x - y\| \geq 2 - \varepsilon\}$.

Daugavet points and Δ-points

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Recall that $\quad \Delta_\varepsilon(x) = \{y \in B_X : \|x - y\| \geq 2 - \varepsilon\}.$

Remark

It is easy to see, that every Daugavet-point is a Δ-point. The reverse is generally not true.

Absolute normalized norm

Let X and Y be Banach spaces.

Definition 3

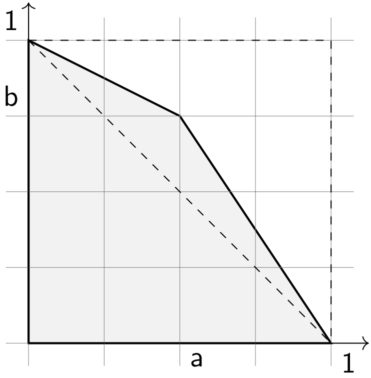
A norm $\|\cdot\|_N$ on $X \times Y$ is said to be *absolute* if there is a function $N\colon [0,\infty) \times [0,\infty) \to [0,\infty)$ such that

$$\|(x,y)\|_N = N(\|x\|, \|y\|) \quad \text{for all } (x,y) \in X \times Y.$$

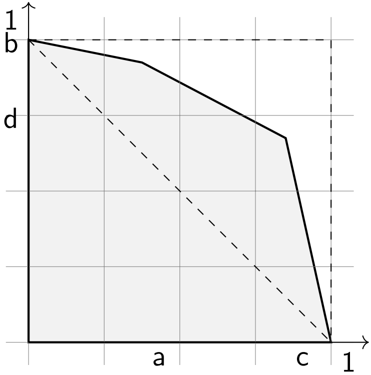
Absolute norm $\|\cdot\|_N$ is normalized if $N(0,1) = N(1,0) = 1$.

Product space $X \times Y$ equipped with an absolute normalized norm $\|\cdot\|_N$ is denoted by $X \oplus_N Y$.

First quadrant of the unit ball of a positively OH norm N



First quadrant of the unit ball of a norm N with property (α)



Results from Abrahamsen, Haller, Lima, P., 2018

Proposition 3

Let X and Y be Banach spaces and N an absolute normalized on $\mathbb{R}^2$. If X and Y have a Δ-point then so does $X \oplus_N Y$.

Results from Abrahamsen, Haller, Lima, P., 2018

Proposition 3

Let X and Y be Banach spaces and N an absolute normalized on $\mathbb{R}^2$. If X and Y have a Δ-point then so does $X \oplus_N Y$.

Proposition 4

Let X and Y be Banach spaces and N a POH norm on $\mathbb{R}^2$. If X and Y have a Daugavet-point then so does $X \oplus_N Y$.

Results from Abrahamsen, Haller, Lima, P., 2018

Proposition 3

Let X and Y be Banach spaces and N an absolute normalized on $\mathbb{R}^2$. If X and Y have a Δ-point then so does $X \oplus_N Y$.

Proposition 4

Let X and Y be Banach spaces and N a POH norm on $\mathbb{R}^2$. If X and Y have a Daugavet-point then so does $X \oplus_N Y$.

Proposition 5

Let X and Y be Banach spaces and N an absolutely nomalized norm with the property (α). Then $X \oplus_N Y$ cannot have any Daugavet-points.

A-octacedral norms

Definition 4

Let X be a Banach space and $A \subset S_X$. We say that an absolute normalized norm N on $\mathbb{R}^2$ is *A-octahedral (A-OH)* if for every $x_1, \ldots, x_n \in A$ and every $\varepsilon > 0$ there exists $y \in S_X$ such that $\|x_i + y\| \geq 2 - \varepsilon$ for every $i \in \{1, \ldots, n\}$.

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Remark

Every POH norm N is a $\{(0,1),(1,0)\}$-octahedral.

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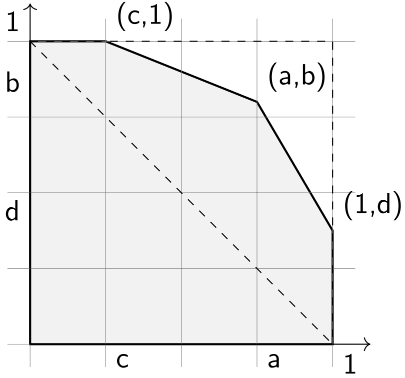
Remark

Every POH norm N is a $\{(0,1),(1,0)\}$-octahedral.

Remark

S_X-octahedrality is octahedrality of a norm in general sense.

First quadrant of the unit ball of a A-OH norm N



A sense of dichotomy of absolute normalized norms

Proposition 6

Let X be a Banach space $c = \max_{N(e,1)=1} e$, $d = \max_{N(1,f)=1} f$ and $A = \{(c,1),(1,d)\}$. The following are equivalent:

(i) N is A-OH,

(ii) N does not have the property (α).

1.

X and Y with Daugavet points

$\Updownarrow$

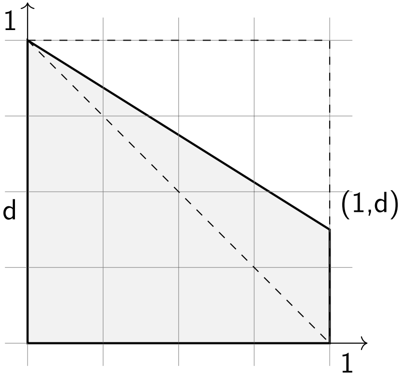
$X \bigoplus_N Y$ with Daugavet points

?

Results concerning Daugavet-points (1)

$N \neq \infty$, $a \neq 0$ and $b \neq 0$	x and y are Daugavet-points $\Leftrightarrow$ (ax, by) is Daugavet-point
$N \neq \infty$ and $a = 0$, $N\big((0,1) + (1,d)\big) = 2$	y is Daugavet-point $\Leftrightarrow$ (ax, by) is Daugavet-point
$N \neq \infty$ ja $b = 0$, $N\big((1,0) + (c,1)\big) = 2$	x is Daugavet-point $\Leftrightarrow$ (ax, by) is Daugavet-point

First quadrant of the unit ball of a special A-OH norm N



Results concerning Daugavet-points (2)

$b = 0$ and $N\big((1,0)+(1,d)\big) < 2$ or $a = 0$ and $N\big((0,1)+(c,1)\big) < 2$	(ax, by) is not Daugavet-point
$N = \infty$	x or y is Daugavet-point $\Leftrightarrow$ (ax, by) is Daugavet-point

2.

$$X \bigoplus_N Y \text{ with } \Delta\text{-points}$$
$$\Updownarrow$$
$$X \text{ and/or } Y \text{ with } \Delta\text{-points}$$
?

Results regarding Δ-points

Theorem 1

Let X and Y be Banach spaces, $x \in S_X$, $y \in S_Y$, N an absolute normalised norm on $\mathbb{R}^2$. Assume that (ax, by) is a Δ-point in $X \oplus_N Y$.

(a) *If $b \neq 1$, then x is a Δ-point in X.*

(b) *If $a \neq 1$, then y is a Δ-point in Y.*

Preparations

Lemma 1 (Abrahamsen, Haller, Lima, P., 2018)

Let X be a Banach space and $x \in S_X$. Then the following assertions are equivalent:

(i) *x is a Δ-point;*

(ii) *for every slice $S(B_X, x^*, \alpha)$ of B_X, with $x \in S(B_X, x^*, \alpha)$, and every $\varepsilon > 0$ there exists $u \in S(B_X, x^*, \alpha)$ such that $\|x - u\| \geq 2 - \varepsilon$.*

Definition 5

Let X be a Banach space, $x \in S_X$, and $k > 1$. We say that x is a Δ_k-point in X, if for every $S(B_X, x^*, \alpha)$ with $x \in S(B_X, x^*, \alpha)$ and for every $\varepsilon > 0$ there exists $u \in S(B_X, x^*, k\alpha)$ such that $\|x - u\| \geq 2 - \varepsilon$.

Δ_k-point need not be Δ-point

Example 1

Let X and Y be Banach spaces, $x \in S_X$ and $y \in S_Y$, and let $k > 1$. Set $Z = X \oplus_1 Y$ and $z = \big((1 - 1/k)x, y/k\big)$. Assume that x is not a Δ-point in X and y is a Δ-point in Y. Then z is not a Δ-point in Z but z is a Δ_k-point in Z.

Results regarding Δ-points continue

Proposition 7

Let X and Y be Banach spaces, $x \in S_X$ and $y \in S_Y$. Let $p, q > 1$ satisfy $1/p + 1/q = 1$.

(a) *If x is a Δ_p-point in X and y is a Δ_q-point in Y, then (x, y) is a Δ-point in $X \oplus_\infty Y$.*

(b) *If x is not a Δ_p-point in X and y is not a Δ_q-point in Y, then (x, y) is not a Δ-point in $X \oplus_\infty Y$.*

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