Locality estimates for complex-time evolution in 1D

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PART I: Setting and problems

Quantum Many Body Problems

- $\mathcal{H}$ complex Hilbert space (finite-dimensional)

  Norm-one vectors $|\psi\rangle \in \mathcal{H}$ are called *states*.
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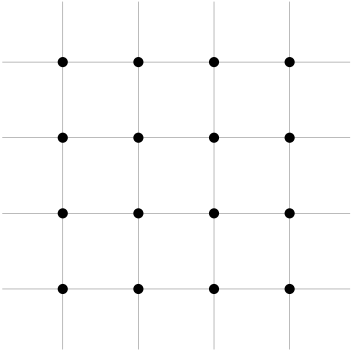
Aim

Study the ground states, i.e. norm-one eigenvectors associated to λ_0.

- Regular lattice, e.g. $\mathbb{Z}^m$.



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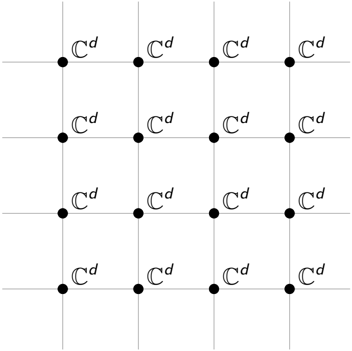

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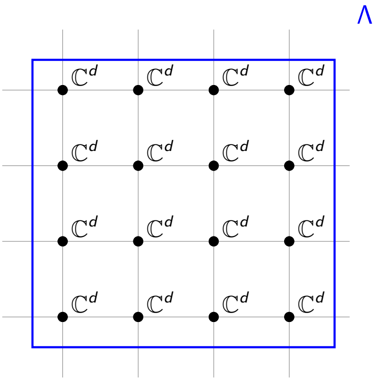

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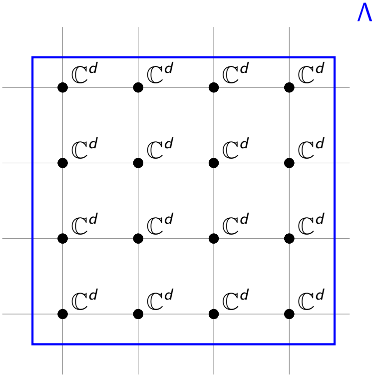
and a local Hamiltonian, e.g.



$$H_\Lambda : \mathcal{H}_\Lambda \to \mathcal{H}_\Lambda \quad , \quad H_\Lambda = \sum_{\substack{x,y\in\Lambda \\ x\sim y}} h_{x,y} \otimes \mathbb{1}_{rest}$$

where $h_{x,y}$ acts on sites $\mathcal{H}_x \otimes \mathcal{H}_y$ as a prefixed self-adjoint operator

$$h : \mathbb{C}^d \otimes \mathbb{C}^d \longrightarrow \mathbb{C}^d \otimes \mathbb{C}^d.$$

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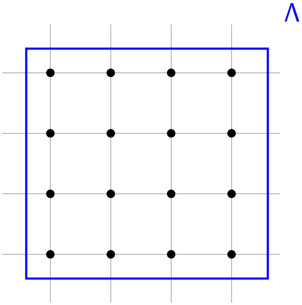

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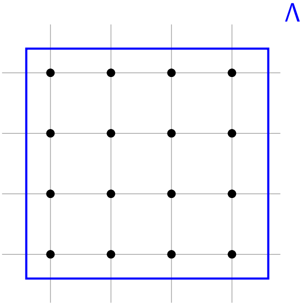

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In general, this problem is **undecidable!!!!**:

- Two dimensions (or greater): Cubitt, Pérez-García, Wolff (Nature, 2015)
- One dimension: Bauch, Cubitt, Lucia, Pérez-García (2018).

Example: AKLT model (1987)

Existence of a spectral gap in the AKLT model on the hexagonal lattice

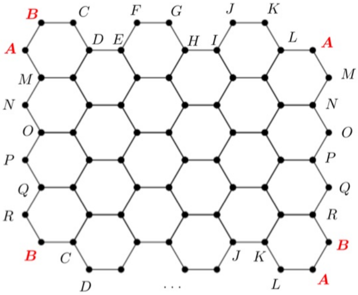
Marius Lemm,[1,*] Anders W. Sandvik,[2,3,†] and Ling Wang[4,‡]
[1] Department of Mathematics, Harvard University, 1 Oxford Street, Cambridge, Massachusetts 02138, USA
[2] Department of Physics, Boston University, 590 Commonwealth Avenue, Boston, Massachusetts 02215, USA
[3] Beijing National Laboratory for Condensed Matter Physics and Institute of Physics, Chinese Academy of Sciences, Beijing 100190, China
[4] Zhejiang Institute of Modern Physics, Zhejiang University, Hangzhou 310027, China
(Dated: October 25, 2019)

$$\mathcal{H}_{m_1,m_2} = \bigotimes_{j\in\Lambda_{m_1,m_2}} \mathbb{C}^4. \qquad (1)$$

On $\mathcal{H}_{m_1,m_2}$, the AKLT Hamiltonian is defined by

$$H^{AKLT}_{m_1,m_2} = \sum_{\substack{j,k\in\Lambda_{m_1,m_2}:\\ j\sim k}} P^{(3)}_{j,k}, \qquad (2)$$

where $P^{(3)}_{j,k}$ denotes the projection onto total spin 3 across the bond connecting vertices j and k. By convention, the



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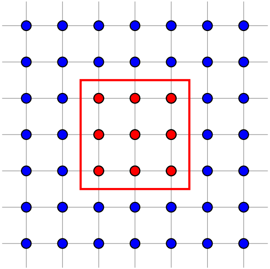
Let $\Lambda = A \cup B \subset \mathbb{Z}^m$ finite set

$$\mathcal{H}_\Lambda = \mathcal{H}_A \otimes \mathcal{H}_B$$

Let $|\psi\rangle$ be a state. It can be written as

$$|\psi\rangle = \sum_j |u_j\rangle \otimes |v_j\rangle.$$



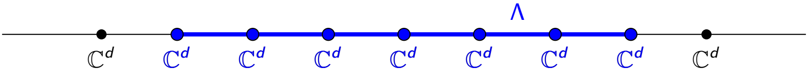
We say that $|\psi\rangle$ is **entangled** if

$$|\psi\rangle \neq |\psi_A\rangle \otimes |\psi_B\rangle \quad \text{for every} \quad |\psi_A\rangle, |\psi_B\rangle.$$

Ground states of local gapped hamiltonians are "simpler" than generic states:

- Low entanglement. Area Law. Decay of Correlations.
- Approximation by Tensor Network States.

Let $\Lambda \subset \mathbb{Z}$ finite set with $N := |\Lambda|$, so that $\mathcal{H}_\Lambda = \otimes_{x\in\Lambda}\mathcal{H}_x \equiv \mathbb{C}^{d^N}$:



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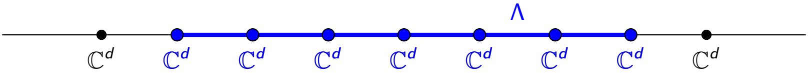
- Generic state (number of parameters: d^N)

$$|\psi\rangle = \sum_{i_1,\ldots,i_N=1}^{d} c_{i_1\ldots i_m} \, |i_1\rangle \otimes \ldots \otimes |i_N\rangle$$



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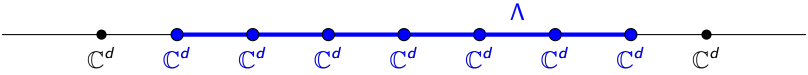
- Tensor networks state (number of parameters: mdD^2)

$$|\psi\rangle = \sum_{i_1,\ldots,i_N=1}^{d} \mathrm{tr}\left(A^{[1]}_{i_1}\ldots A^{[N]}_{i_N}\right)|i_1\rangle \otimes \ldots \otimes |i_N\rangle$$

where $A^{[j]}_i \in M_{D\times D}(\mathbb{C})$ for $i = 1,\ldots,d$ and $j = 1,\ldots,N$.



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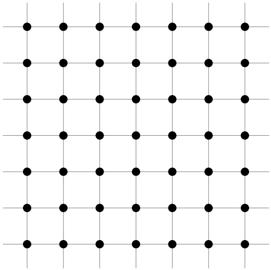
Every tensor network state can be seen as the ground state of a *hamiltonian* called **parent hamiltonian**.

Q. When is this parent Hamiltonian gapped?
In 1D this is known to be a generic property. But what about 2D???

PART II: Locality estimates



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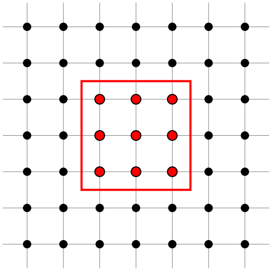

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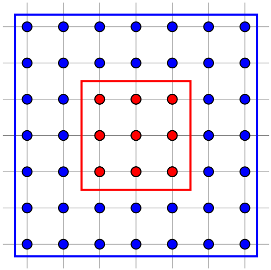

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If $\Lambda \subset \Lambda' \subset \mathbb{Z}^n$, there is a canonical linear isometry

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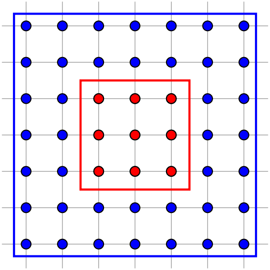

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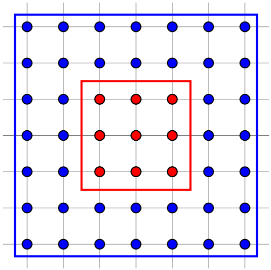

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This allows to consider the algebra of local observables and the algebra of quasi-local observables (its completion)

$$\mathcal{A}_{loc} = \bigcup_{X \text{ finite}} \mathcal{A}_X \quad , \quad \mathcal{A} = \overline{\mathcal{A}_{loc}}$$

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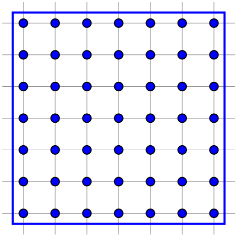

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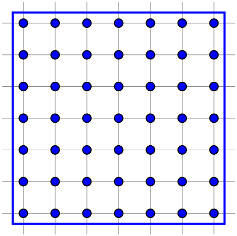

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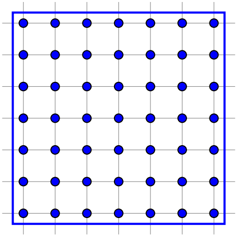

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Using dynamics:

$$\begin{array}{ccccc} \mathbb{R} \ni t & \mapsto & \Gamma^t_\Lambda : \mathcal{A} & \longrightarrow & \mathcal{A} \\ & & A & \mapsto & e^{itH_\Lambda} A e^{-itH_\Lambda} \end{array}$$

Maybe we can define

$$\mathbb{R} \ni t \quad \mapsto \quad \Gamma^t(A) = \lim_{\Lambda\nearrow\mathbb{Z}^d} \Gamma^t_\Lambda(A) \quad ????$$

Lieb-Robinson estimates

Let Φ be interaction on $\mathbb{Z}^m$ with

$$X \text{ finite} \quad \mapsto \quad \Phi_X \in \mathcal{A}_X \quad \text{s.t.} \quad \|\Phi_X\| = O(e^{-\lambda\, \mathrm{diam}(X)})$$

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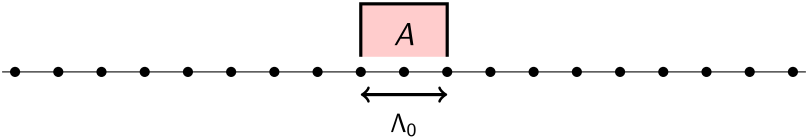

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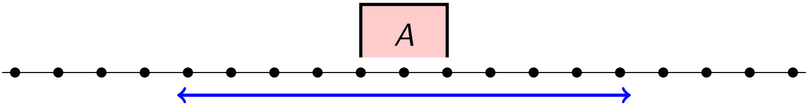
Then, for every $0 \leq \ell \leq L$

$$\left\|\Gamma^t_{\Lambda_L}(A) - \Gamma^t_{\Lambda_\ell}(A)\right\| \;\leq\; C\,|\Lambda_0|\,\|A\|\,e^{-\mu(\ell - v|t|)}$$



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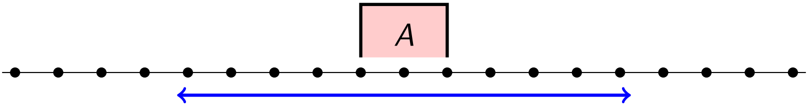

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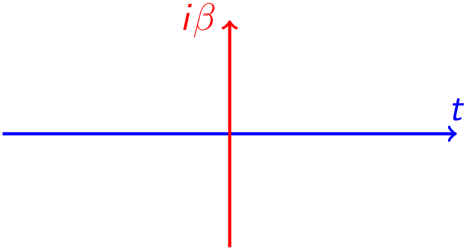
Lieb, Robinson (1972), Hastings (2004), Nachtergaele, Sims (2006), Hastings, Koma (2006), Nachtergaele, Ogata, Sims (2010) . . .

Problem

Let Φ interaction with exponential decay and let $A \in \mathcal{A}_{\Lambda_0}$ and let

$$\Gamma^s(A) = e^{isH} A \, e^{-isH}, \quad s \in \mathbb{C}.$$

$$\left\| \Gamma^s_{\Lambda_L}(A) - \Gamma^s_{\Lambda_\ell}(A) \right\| \leq \quad ???$$



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converges exponentially fast over compact subsets of the region:

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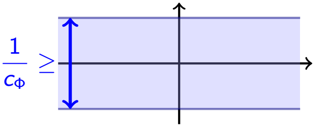
Finite Range $\Phi_X = 0$ if $\operatorname{diam}(X) \geq r$	$\mathbb{C}$ Araki 1969	 Robinson 68, Ruelle 69 Bouch 2015
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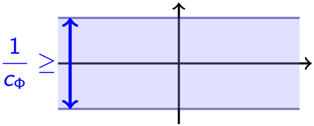
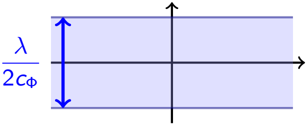
Finite Range $\Phi_X = 0$ if $\operatorname{diam}(X) \geq r$	$\mathbb{C}$ Araki 1969	 Robinson 68, Ruelle 69 Bouch 2015
Exponential Decay $\\|\Phi_X\\| = O(e^{-\lambda \operatorname{diam}(X)})$	 Optimal???	

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Mathematical open problems in projected entangled pair states

Juan Ignacio Cirac[1] · José Garre-Rubio[2,3] · David Pérez-García[2,3]

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Abstract
Projected entangled pair states (PEPS) are used in practice as an efficient parametrization of the set of ground states of quantum many body systems. The aim of this paper is to present, for a broad mathematical audience, some mathematical questions about PEPS.