ON THICKNESS-LIKE INDICES OF BANACH SPACES

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Introduction

This talk is based on the joint work with R. Haller, J. Langemets and V. Lima.

We only look at non-trivial real Banach spaces. Throughout, let X, Y be Banach spaces.

A slice of B_X is a set of the form

$$S(x^*, \alpha) = \{x \in B_X \mid x^*(x) > 1 - \alpha\},$$

where $x^* \in S_{X^*}$ and $\alpha > 0$.

By a convex combination of sets $S_1, \dots, S_n$ we mean the set

$$\sum_{i=1}^{n} \lambda_i S_i,$$

where $\lambda_i > 0$ and $\lambda_1 + \cdots + \lambda_n = 1$.

Diameter 2 properties

In [ALN] T. Abrahamsen, V. Lima and O. Nygaard introduced the diameter 2 properties.

Definition

We say that X has

1. *the local diameter 2 property (LD2P) if every slice of the unit ball B_X has diameter 2;*
2. *the diameter 2 property (D2P) if every relatively weakly open subset of the unit ball B_X has diameter 2;*
3. *the strong diameter 2 property (SD2P) if every convex combination of slices of the unit ball B_X has diameter 2.*

The Daugavet property

In [D] I. Daugavet introduces the following property

Definition

A Banach space X has the Daugavet property (DP) if for every rank-1 operator $T : X \to X$

$$\|I + T\| = 1 + \|T\|.$$

Almost a picture!

The following relations hold.

$$DP \Rightarrow SD2P \Rightarrow D2P \Rightarrow LD2P$$

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However, the converse implications do not hold.

Whitley's thickness index

In [W] R. Whitley defined the following index

$$T_W(X) := \inf \left\{ r > 0 \mid \exists x_1, \ldots, x_n \in S_X \text{ so that } S_X \subset \bigcup_{i=1}^{n} B(x_i, r) \right\}.$$

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In [CPS] it was shown that $T_W(X)$ is equal to

$$T(X) := \inf \left\{ r > 0 \mid \exists x_1, \ldots, x_n \in S_X \text{ so that } B_X \subset \bigcup_{i=1}^{n} B(x_i, r) \right\}$$

when X is a infinite-dimensional space.

Daugavet index

In [RZ] A. Rueda Zoca introduced the following index

$$\mathcal{T}(X) := \inf \Big\{ r > 0 \,|\exists x \in S_X \exists \emptyset \neq W \subset B_X \text{ weakly open} \\ \text{such that } W \subset B(x,r) \Big\}$$

and its dual counterpart

$$\mathcal{T}_{w^*}(X) := \inf \Big\{ r > 0 \,|\exists x \in S_X \exists \emptyset \neq W \subset B_X \text{ weak}^* \text{ open} \\ \text{such that } W \subset B(x,r) \Big\}$$

New Daugavet indices

$$\mathcal{T}^s(X) = \inf \left\{ r > 0 \mid \begin{array}{l} \exists x \in S_X \text{ and a slice } S \text{ of } B_X \\ \text{such that } S \subset B(x, r) \end{array} \right\}.$$

$$\mathcal{T}^{ccs}(X) = \inf \left\{ r > 0 \mid \begin{array}{l} \exists\, x \in S_X \text{ and } C \neq \emptyset \text{ a convex combination} \\ \text{of slices of } B_X \text{ such that } C \subset B(x, r) \end{array} \right\}.$$

$$\mathcal{T}^{ccw}(X) = \inf \left\{ r > 0 \mid \begin{array}{l} \exists x \in S_X \text{ and } C \neq \emptyset \text{ a convex combination} \\ \text{of relatively weakly open subsets of } B_X \\ \text{such that } C \subset B(x, r) \end{array} \right\}.$$

Since $\mathcal{T}^{ccs}(X) = \mathcal{T}^{ccw}(X)$, we will rename it as $\mathcal{T}^{cc}(X)$.

Properties of Daugavet indices

It is not difficult to see that

$$2 \geq \mathcal{T}^s(X) \geq \mathcal{T}(X) \geq \mathcal{T}^{cc}(X) \geq 0.$$

1. If X has the LD2P, then $\mathcal{T}^s(X) \geq 1$.
2. If X has the D2P, then $\mathcal{T}(X) \geq 1$.
3. If X has the SD2P, then $\mathcal{T}^{cc}(X) \geq 1$.

Finding spaces

If $1 < p < \infty$ then $\mathcal{T}^s(X \oplus_p Y) \leq 2^{1/p}$ and if X, Y have DP then $\mathcal{T}(X \oplus_p Y) \geq 2^{1/p}$.
Therefore for each $\delta \in [1, 2]$ there exists a space X such that $\mathcal{T}^s(X) = \mathcal{T}(X) = \delta$.
Namely,

$$\mathcal{T}^s(c_0) = \mathcal{T}(c_0) = 1$$
$$\mathcal{T}^s(C[0,1] \oplus_p C[0,1]) = \mathcal{T}(C[0,1] \oplus_p C[0,1]) = 2^{1/p}$$
$$\mathcal{T}^s(C[0,1]) = \mathcal{T}(C[0,1]) = 2$$

A similarity with the SD2P

It is known that when $1 < p < \infty$ then $X \oplus_p Y$ does not have the SD2P.

In [HLN] it was shown that for $1 < p < \infty$ and X, Y with SD2P the space $X \oplus_p Y$ has the SDδP where $\delta = 2^{1-1/p}$.

For the convex Daugavet index we have that $\mathcal{T}^{cc}(c_0 \oplus_p c_0) = 2^{-1/p}$.

Rueda Zoca showed that for every norm 1 and weakly compact operator $T : X \to X$, it follows that

$$\|I + T\| \geq \max\{\mathcal{T}(X), \mathcal{T}_{w^*}(X^*)\}.$$

and asked whether or not

$$\begin{aligned} \inf\{\|T + I\| \mid T \in S_{\mathcal{L}(X,X)} \text{ and } T \text{ is weakly compact}\} \\ = \max\{\mathcal{T}(X), \mathcal{T}_{w^*}(X)\}. \end{aligned}$$

The answer is no, because [BGLPRZ] gave an example of a space with the LD2P but with r.w.o. sets with arbitrarily small diameter.

Thank you for your attention!

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Thank you for your attention (again)!

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