# Analytic structure in fibers of the space of bounded analytic functions on the unit ball of $c_0$

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Joint work with Yun Sung Choi, Javier Falcó, Domingo García, & Manuel Maestre
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Around 1955, S. Kakutani was the first one to study the space

$$\mathscr{H}^\infty(\mathbb{D}) = \{f: \mathbb{D} \to \mathbb{C} \text{ holomorphic \& bounded} \}$$

as a 'Banach algebra'.

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We call  $\mathscr{M}(\mathscr{H}^\infty(\mathbb{D}))$  the **spectrum** or **maximal ideal space** of  $\mathscr{H}^\infty(\mathbb{D})$ 

For  $z \in \mathbb{D}$ , let us consider the *point evaluation* at z

$$\delta(z)(f) := f(z) \qquad (f \in \mathcal{H}^{\infty}(\mathbb{D}))$$

The set  $\mathscr{M}(\mathscr{H}^\infty(\mathbb{D}))\setminus\overline{\{\delta(z):z\in\mathbb{D}\}}^{w^*}$  is referred to as the **Corona**...

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## Corona Theorem (L. Carleson, 1962)

$$\mathscr{M}(\mathscr{H}^{\infty}(\mathbb{D})) = \overline{\{\delta(z) : z \in \mathbb{D}\}}^{w^*}$$

i.e., the Corona set is empty.

On the other hand, there is a natural surjective map

$$\pi: \mathcal{M}(\mathcal{H}^{\infty}(\mathbb{D})) \longrightarrow \overline{\mathbb{D}}$$
$$\varphi \longmapsto \varphi(z \mapsto z)$$

The **fiber** of the spectrum  $\mathscr{M}(\mathscr{H}^\infty(\mathbb{D}))$  at the point  $z\in\mathbb{D}$  is defined as

$$\mathcal{M}_{z}(\mathcal{H}^{\infty}(\mathbb{D})) = \{ \psi \in \mathcal{M}(\mathcal{H}^{\infty}(\mathbb{D})) : \pi(\psi) = z \} = \pi^{-1}(z)$$

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Given a complex Banach space X, we consider

$$\mathscr{H}^{\infty}(B_X) = \{ f : B_X \to \mathbb{C} \text{ holomorphic \& bounded} \}$$

and its spectrum

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endowed with the weak-star topology. There is a natural inclusion map

$$\delta: B_{X^{**}} \longrightarrow \mathcal{M}(\mathcal{H}^{\infty}(B_X))$$
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where  $\delta(z)(f):= ilde{f}(z)$  for  $f\in\mathscr{A}$  , where  $ilde{f}$  is the Aron-Berner extension of f .



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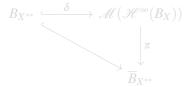


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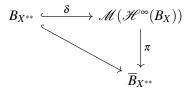


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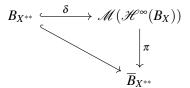


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## When X is finite dimensional, then

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 for every  $z\in B_X.$ 

In other words, the fiber at a point z lying inside  $B_X$  is just a singleton set

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I. J. Schark = I. Kaplansky + J. Wermer + S. Kakutani + C. Buck + H. Royden + A. Gleason + R. Arens + K. Hoffman, 1961

There exists an analytic map  $\Psi : \mathbb{D} \to \mathcal{M}(\mathcal{H}^{\infty}(\mathbb{D}))$  which is a homeomorphism and actually maps  $\mathbb{D}$  into the fiber  $\mathcal{M}_1(\mathcal{H}^{\infty}(\mathbb{D}))$ .

Recall that  $\mathcal{M}_z(\mathscr{H}^\infty(B_X))=\{\delta_z\}$  for  $z\in B_X$  when X is finite dimensional.

The situation for **infinite dimensional** X is quite different

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• Suppose X is infinite dimensional. Then the fiber  $\mathcal{M}_z(\mathcal{H}^\infty(B_X))$  over any  $z\in \overline{B}_{X^{**}}$  contains a copy of  $\beta\mathbb{N}\setminus\mathbb{N}$ .

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#### R. M. Aron, J. Falcó, D. García, M. Maestre, 2018

$$B_{\ell_\infty} \stackrel{\mathit{analytically}}{\longleftrightarrow} \mathscr{M}_{\mathcal{Z}}(\mathscr{H}^\infty(B_{c_0}))$$

where  $z=(z_1,z_2,\dots)$  be a point of the distinguished boundary  $\mathbb{T}^{\aleph_0}$  of  $\overline{B}_{\ell_\infty}$  (i.e.,  $|z_j|=1$  for all  $j\in\mathbb{N}$ ).

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**Question**: Can  $B_{\ell_{\infty}}$  be embedded in the fiber  $\mathcal{M}_z(\mathcal{H}^{\infty}(B_{c_0}))$  when  $z=(z_n)$  is in the unit sphere of  $\ell_{\infty}$  but  $|z_n|<1$  for all  $n\in\mathbb{N}$ , for example,  $z_n=\frac{n-1}{n}$ ?

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#### Choi-Falcó-García-J-Maestre, 2019

For every natural number N and every  $z \in \overline{B}_{\ell_m}$ ,

$$\underbrace{B_{\ell_\infty} \times \cdots \times B_{\ell_\infty}}_{N\text{-times}} \overset{\textit{analytically}}{\subseteq} \mathscr{M}_{\mathbf{Z}}(\mathscr{H}^\infty(B_{c_0})).$$

#### Choi-Falcó-García-J-Maestre, 2019

If K is countably infinite scattered compact Hausdorff,

$$B_{\ell_\infty} \stackrel{\mathit{analytically}}{\longleftrightarrow} \mathscr{M}_{\mathit{u}}(\mathscr{H}^\infty(B_{C(K)})).$$

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Suppose X has a normalized shrinking basis  $\{e_j\}$ . Suppose that there is an integer  $N \geq 1$  such that

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**Question 0**: Does the Corona Theorem hold in  $\mathscr{H}^{\infty}(U)$  for any domain U of  $\mathbb{C}$ ?

**Question 1**: Does the Corona Theorem hold in  $\mathscr{H}^\infty(\mathbb{D}^n)$  or  $\mathscr{H}^\infty(B_{\ell_2^n})$  for any  $n \geq 2$ ?

**Question 2**: From the last result, we have that for  $1 and <math>z \in B_{\ell_p}$ 

$$B_{\ell_\infty} \stackrel{\mathit{analytically}}{\longleftrightarrow} \mathscr{M}_{\mathbf{Z}}(\mathscr{H}^\infty(B_{\ell_p})).$$

Can  $B_{\ell_\infty}$  be embedded in the fiber  $\mathscr{M}_z(\mathscr{H}^\infty(B_{\ell_p}))$  when  $z\in S_{\ell_p}$ ?

**Question 3**: Can  $B_{\ell_{\infty}}$  be embedded in the fiber  $\mathscr{M}_z(\mathscr{H}^{\infty}(B_{\ell_1}))$  when  $z\in \overline{B}_{\ell_1^{**}}$ ?

