Analytic structure in fibers of the space of bounded analytic functions on the unit ball of c_0

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Joint work with Yun Sung Choi, Javier Falcó, Domingo García, & Manuel Maestre

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Around 1955, S. Kakutani was the first one to study the space

$$\mathscr{H}^\infty(\mathbb{D}) = \{f : \mathbb{D} \to \mathbb{C} \text{ holomorphic } \& \text{ bounded}\}$$

as a *'Banach algebra'*.

As we consider the space $\mathscr{H}^\infty(\mathbb{D})$ as a Banach algebra...

$$\mathscr{M}(\mathscr{H}^\infty(\mathbb{D})) = \{\varphi : \mathscr{H}^\infty(\mathbb{D}) \to \mathbb{C} \text{ non-zero homomorphisms}\}$$

We call $\mathscr{M}(\mathscr{H}^\infty(\mathbb{D}))$ the **spectrum** or **maximal ideal space** of $\mathscr{H}^\infty(\mathbb{D})$.

For $z \in \mathbb{D}$, let us consider the *point evaluation* at z:

$$\delta(z)(f) := f(z) \qquad (f \in \mathscr{H}^\infty(\mathbb{D}))$$

The set $\mathscr{M}(\mathscr{H}^\infty(\mathbb{D})) \setminus \overline{\{\delta(z) : z \in \mathbb{D}\}}^{w^*}$ is referred to as the **Corona**...

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Corona Theorem (L. Carleson, 1962)

$$\mathscr{M}(\mathscr{H}^\infty(\mathbb{D})) = \overline{\{\delta(z) : z \in \mathbb{D}\}}^{w^*}$$

i.e., the Corona set is empty.

On the other hand, there is a natural surjective map

$$\pi : \mathscr{M}(\mathscr{H}^\infty(\mathbb{D})) \longrightarrow \overline{\mathbb{D}}$$

$$\varphi \longmapsto \varphi(z \mapsto z)$$

The **fiber** of the spectrum $\mathscr{M}(\mathscr{H}^\infty(\mathbb{D}))$ at the point $z \in \overline{\mathbb{D}}$ is defined as

$$\mathscr{M}_z(\mathscr{H}^\infty(\mathbb{D})) = \{\psi \in \mathscr{M}(\mathscr{H}^\infty(\mathbb{D})) : \pi(\psi) = z\} = \pi^{-1}(z).$$

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Given a complex Banach space X, we consider

$$\mathscr{H}^\infty(B_X) = \{f : B_X \to \mathbb{C} \text{ holomorphic } \& \text{ bounded}\}$$

and its **spectrum**

$$\mathscr{M}(\mathscr{H}^\infty(B_X)) = \{\varphi : \mathscr{H}^\infty(B_X) \to \mathbb{C} \text{ non-zero homomorphisms}\}$$

endowed with the weak-star topology. There is a natural inclusion map

$$\delta : B_{X^{**}} \longrightarrow \mathscr{M}(\mathscr{H}^\infty(B_X))$$
$$z \longmapsto \delta(z),$$

where $\delta(z)(f) := \tilde{f}(z)$ for $f \in \mathscr{A}$, where $\tilde{f}$ is the Aron-Berner extension of f.

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where $\delta(z)(f) := \tilde{f}(z)$ for $f \in \mathscr{A}$, where $\tilde{f}$ is the Aron-Berner extension of f.

Also, there is a natural surjective map

$$\begin{aligned} \pi : \mathscr{M}(\mathscr{H}^\infty(B_X)) &\longrightarrow \overline{B}_{X^{**}} \\ \psi &\longmapsto \psi|_{X^*} \end{aligned}$$

We have the following commutative diagram:

$$\begin{array}{ccc} B_{X^{**}} & \overset{\delta}{\hookrightarrow} & \mathscr{M}(\mathscr{H}^\infty(B_X)) \\ & \searrow & \downarrow \pi \\ & & \overline{B}_{X^{**}} \end{array}$$

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When X is finite dimensional, then

$$\mathscr{M}_z(\mathscr{H}^\infty(B_X)) = \{\delta_z\} \text{ for every } z \in B_X.$$

In other words, the fiber at a point z lying inside B_X is just a singleton set.

*Surprisingly, what happens with the fibers of $\mathscr{M}(\mathscr{H}^\infty(B_X))$ **at a boundary point $z \in S_X$** is in totally different situation.*

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I. J. Schark* $=$ *I. Kaplansky* $+$ *J. Wermer* $+$ *S. Kakutani* $+$ *C. Buck* $+$ *H. Royden* $+$ *A. Gleason* $+$ *R. Arens* $+$ *K. Hoffman, 1961

There exists an analytic map $\Psi : \mathbb{D} \to \mathscr{M}(\mathscr{H}^\infty(\mathbb{D}))$ which is a homeomorphism and actually maps $\mathbb{D}$ into the fiber $\mathscr{M}_1(\mathscr{H}^\infty(\mathbb{D}))$.

Recall that $\mathscr{M}_z(\mathscr{H}^\infty(B_X)) = \{\delta_z\}$ *for* $z \in B_X$ *when* X *is finite dimensional.*

The situation for ***infinite dimensional*** X *is quite different.*

R. M. Aron, B. J. Cole, T. W. Gamelin, 1991

- Suppose X is infinite dimensional. Then the fiber $\mathscr{M}_z(\mathscr{H}^\infty(B_X))$ over any $z \in \overline{B}_{X^{**}}$ contains a copy of $\beta\mathbb{N} \setminus \mathbb{N}$.

- $B_{\ell_\infty} \xhookrightarrow[\text{Gleason isometrically}]{\text{analytically}} \mathscr{M}_0(\mathscr{H}^\infty(B_{c_0}))$.

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R. M. Aron, J. Falcó, D. García, M. Maestre, 2018

$$B_{\ell_\infty} \xhookrightarrow{\text{analytically}} \mathscr{M}_z(\mathscr{H}^\infty(B_{c_0}))$$

where $z = (z_1, z_2, \dots)$ be a point of the distinguished boundary $\mathbb{T}^{\aleph_0}$ of $\overline{B}_{\ell_\infty}$ (i.e., $|z_j| = 1$ for all $j \in \mathbb{N}$).

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Question: Can B_{ℓ_∞} be embedded in the fiber $\mathscr{M}_z(\mathscr{H}^\infty(B_{c_0}))$ when $z = (z_n)$ is in the unit sphere of ℓ_∞ but $|z_n| < 1$ for all $n \in \mathbb{N}$, for example, $z_n = \frac{n-1}{n}$?

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Choi–Falcó–García–J–Maestre, 2019

For every natural number N and **every** $z \in \overline{B}_{\ell_\infty}$,

$$\underbrace{B_{\ell_\infty} \times \cdots \times B_{\ell_\infty}}_{N\text{-times}} \xhookrightarrow[\text{Gleason isometrically}]{\text{analytically}} \mathscr{M}_z(\mathscr{H}^\infty(B_{c_0})).$$

Choi–Falcó–García–J–Maestre, 2019

If K is countably infinite scattered compact Hausdorff,

$$B_{\ell_\infty} \xhookrightarrow{\text{analytically}} \mathcal{M}_u(\mathcal{H}^\infty(B_{C(K)})).$$

for every $z \in \overline{B}_{C(K)^{**}}$,

B. J. Cole, T. W. Gamelin, W. B. Johnson, 1991

Suppose X has a normalized shrinking basis $\{e_j\}$. Suppose that there is an integer $N \geq 1$ such that

$$\sum_j \left| e_j^*(x) \right|^N < \infty$$

for all $x = \sum_j e_j^*(x) e_j$ in X. Then

$$B_{\ell_\infty} \xhookrightarrow{\text{analytically}} \mathscr{M}_0(\mathscr{H}^\infty(B_X)).$$

Note that this result applies to the spaces ℓ_p and $L_p[0,1]$ for $1 < p < \infty$.

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- **for any $z \in B_X$,**

$$B_{\ell_\infty} \overset{\text{analytically}}{\hookrightarrow} \mathscr{M}_z(\mathscr{H}^\infty(B_X)).$$

- if X has a normalized shrinking **monotone** basis, then **for any $z \in B_{X^{**}}$,**

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Choi–Falcó–García–J–Maestre, 2019

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Question 0: Does the Corona Theorem hold in $\mathscr{H}^\infty(U)$ for any domain U of $\mathbb{C}$?

Question 1: Does the Corona Theorem hold in $\mathscr{H}^\infty(\mathbb{D}^n)$ or $\mathscr{H}^\infty(B_{\ell_2^n})$ for any $n \geq 2$?

Question 2: From the last result, we have that for $1 < p < \infty$ and $z \in B_{\ell_p}$

$$B_{\ell_\infty} \xhookrightarrow{\text{analytically}} \mathscr{M}_z(\mathscr{H}^\infty(B_{\ell_p})).$$

Can B_{ℓ_∞} be embedded in the fiber $\mathscr{M}_z(\mathscr{H}^\infty(B_{\ell_p}))$ when $z \in S_{\ell_p}$?

Question 3: Can B_{ℓ_∞} be embedded in the fiber $\mathscr{M}_z(\mathscr{H}^\infty(B_{\ell_1}))$ when $z \in \overline{B}_{\ell_1^{**}}$?

♡ Yo te quiero... Gracias! ♡