Describing multiplicative convex functions

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Convex functions

Definition

A function $f : V \to \mathbb{R}$ (where V is a vector space over $\mathbb{R}$) is called **convex** if, whenever $x, y \in V$ and $0 \le \lambda \le 1$, we have

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Definition (Niculescu, 2000)

Let $I \subseteq (0, \infty)$ be an interval. A function $f : I \to (0, \infty)$ is called **multiplicative convex** if, for every $x, y \in I$ and $\lambda \in [0, 1]$, we have

$$f(x^{1-\lambda} y^{\lambda}) \le f(x)^{1-\lambda} f(y)^{\lambda}.$$

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Definition

Let $f : (0, \infty) \to [0, \infty)$ be such that $f(1) = 1$. We will say that f is **multiplicative convex** if, for every $\mu > 0$ and $x, y > 0$ we have

$$f(x^{\mu} y^{1/\mu}) \leq f(x)^{\mu} f(y)^{1/\mu}. \tag{1}$$

In particular, by setting $\mu = 1$, we obtain $f(xy) \leq f(x)f(y)$ for every $x, y \geq 0$.

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On the other hand,

$$f(x) = f((x^n)^{\frac{1}{n}} \cdot 1^n) \leq f(x^n)^{\frac{1}{n}} f(1)^n = f(x^n)^{\frac{1}{n}},$$

so that, $f(x)^n \leq f(x^n) \leq f(x)^n$ and therefore $f(x)^n = f(x)^n$.

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For the general case,

$$f(x^q) = f(x^{\frac{n}{m}}) = f(x^{\frac{1}{m}})^n = f(x)^{\frac{n}{m}} = f(x)^q.$$

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Let f be a multiplicative convex function. Then, $f(x^t) = f(x)^t$ for every $t > 0$.

Characterization

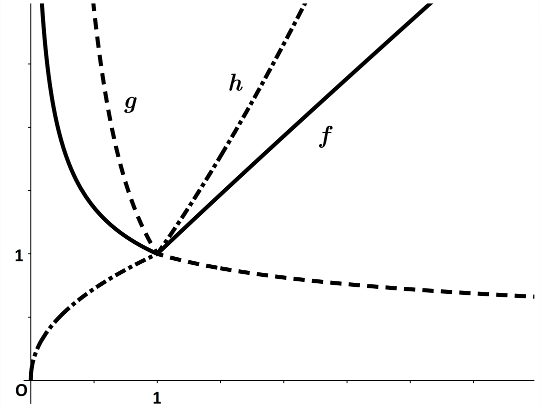
Theorem

Let $f : (0, \infty) \to [0, \infty)$*. Then,* f *is a multiplicative convex function if and only if it can be written of the form*

$$f(x) = \begin{cases} b^{\log_a(x)} & \text{if } 0 < x \le 1, \\ b'^{\log_{a'}(x)} & \text{if } x > 1, \end{cases} \tag{2}$$

where a, b, a' *and* b' *satisfy the following conditions:*

1. $0 < a < 1$ *and* $a' > 1$*.*
2. *If* $b < 1$*, then* $\log_b(b') \le \log_a(a') < 0$ *(which, in particular, implies* $b' > 1$*).*
3. *If* $b > 1$*, then* $\log_b(b') \ge \log_a(a')$*.*



The functions are defined as in Equation (2) by using the following constants: $a_f = 0{,}27$, $b_f = 1{,}8$, $a'_f = 2{,}35$, $b'_f = 2{,}25$, $a_g = 0{,}45$, $b_g = 3{,}45$, $a'_g = 2{,}62$, $b'_g = 0{,}75$, $a_h = 0{,}18$, $b_h = 0{,}45$, $a'_h = 1{,}86$, and $b'_h = 2{,}4$

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If f verifies (1), then $f(1) = f(1 \cdot 1) \leq f(1)^2 \rightsquigarrow f(1) \geq 1$.

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If a function f satisfies (1) would we have that $g(x) = f(x)/f(1)$ verifies (1)?

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Taking $x = y = \frac{1}{2}$ and $\mu = 10$, we have that,
on the one hand,

$$g(x^{\mu} y^{1/\mu}) = \frac{x^{\mu} y^{1/\mu} + 1}{2} = \frac{\left(\frac{1}{2}\right)^{10+1/10} + 1}{2} \approx 0{,}50045,$$

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and, on the other hand,

$$g(x)^{\mu} g(y)^{1/\mu} = \left(\frac{x+1}{2}\right)^{\mu} \left(\frac{y+1}{2}\right)^{1/\mu} = \left(\frac{3}{4}\right)^{10+1/10} \approx 0{,}05471.$$

1. Jiménez Rodríguez; P., Muñoz–Fernánez; G.A., Martínez–Gómez, M. Elena; Seoane–Sepúlveda, Juan B. "Describing multiplicative convex functions". **Journal of Convex Analysis**. Volumen 27, año 2020, No. 2.

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