Optimal Polynomial Approximation in Function Spaces

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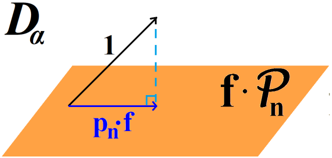
Joint work with C. Bénéteau, O. Ivrii, D. Seco



V Congreso de Jóvenes Investigadores de la RSME
29 January 2020

Three classical function spaces on the unit disc $\mathbb{D}$

- the *Bergman space* A^2, consisting of all functions $f \in \text{Hol}(\mathbb{D})$ with
$$\int_{\mathbb{D}} |f(z)|^2 dA(z) < \infty, \quad dA(z) = \frac{dxdy}{\pi},$$

- the *Hardy space* H^2, consisting of all functions $f \in \text{Hol}(\mathbb{D})$ with
$$\sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta < \infty,$$

- the *Dirichlet space* D, consisting of all functions $f \in \text{Hol}(\mathbb{D})$ with
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Dirichlet-type spaces D_α

The spaces A^2, H^2, and D belong to the broad family of "Dirichlet-type spaces D_α"(for $\alpha = -1, 0, 1$ respectively):

- **Definition:** For $\alpha \in \mathbb{R}$, the space D_α consists of all functions $f \in \mathrm{Hol}(\mathbb{D})$ whose Taylor coefficients in the expansion

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \quad z \in \mathbb{D},$$

satisfy

$$\|f\|_\alpha^2 = \sum_{k=0}^{\infty} (k+1)^\alpha |a_k|^2 < \infty.$$

- For two functions $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$ in D_α, by considering the associated inner product

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Cyclicity

- A function $f \in D_\alpha$ is said to be **cyclic** in D_α if

$$\overline{\mathrm{span}\{z^k f : k = 0, 1, 2, \dots\}} = D_\alpha$$

or equivalently:
if there exists a sequence of polynomials $\{p_n\}_{n=1}^{\infty}$ such that

$$\|p_n f - 1\|_\alpha \to 0, \quad \text{as } n \to \infty.$$

Remarks:

- f is cyclic $\Rightarrow$ f is zero-free on $\mathbb{D}$.
- For H^2 (Beurling): f is cyclic $\Leftrightarrow$ f is outer.
- For D (Brown and Shields): f is cyclic $\Rightarrow$ f is outer **and** $\{\zeta \in \partial\mathbb{D} : \lim_{r \to 1^-} f(r\zeta) = 0\}$ is of logarithmic capacity 0.
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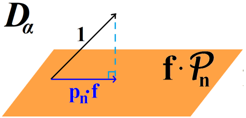
Optimal Polynomial Approximants (O.P.A.)

Let $f \in D_\alpha$ and $\mathcal{P}_n$ be the space of polynomials of degree $\leq n$.

Definition

We say that a polynomial $p_n \in \mathcal{P}_n$ is an **optimal polynomial approximant** (o.p.a.) of order n to $1/f$ if p_n minimizes $\|pf - 1\|_\alpha$ among all polynomials $p \in \mathcal{P}_n$.



- **Remark:**

$$\|p_n f - 1\|_\alpha = \text{dist}_{D_\alpha}(1, f \cdot \mathcal{P}_n) \qquad \text{and}$$

$p_n f$ is the orthogonal projection of 1 onto the subspace $f \cdot \mathcal{P}_n$. Thus, for any $f \in D_\alpha \setminus \{0\}$ and any degree $n \geq 0$, the o.p.a. p_n to $1/f$ always **exist** and are **unique**.

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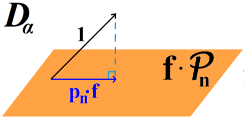

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O.P.A. and Cyclicity

♣ **Notation:** Given $f \in D_\alpha \setminus \{0\}$, let $Q_n(1/f)$ denote the o.p.a. of order n to $1/f$.

- **Remark:** T.F.A.E.

1. f is cyclic in D_α.
2. $\|Q_n(1/f) \cdot f - 1\|_\alpha \to 0$.
3. $Q_n(1/f) \to 1/f$ uniformly on compact subsets of $\mathbb{D}$.
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- **Question:** **Which is the behaviour of the sequence $(Q_n(1/f))$ on the unit circle?**

or

In this talk we will focus on o.p.a. for the **Hardy space** H^2.

- If f is inner (i.e. f bounded on $\mathbb{D}$ and $|\lim_{r\to 1^-} f(r\zeta)| = 1$ for a.e. $\zeta \in \partial\mathbb{D}$) then $Q_n(1/f) = \overline{f(0)}$ for each $n \in \mathbb{N}$.

- If f is holomorphic and zero-free on $\{z : |z| < 1 + \varepsilon\}$ for some $\varepsilon > 0$ then, for each ζ in the unit circle, $Q_n(1/f)(\zeta) \to 1/f(\zeta)$ as $n \to \infty$.

- **(Bénéteau, M., Seco)** If f is a polynomial with only simple roots, all of which lying outside $\mathbb{D}$, then $Q_n(1/f) \to 1/f$ uniformly on compact subsets of $\overline{\mathbb{D}} \setminus \{z \in \partial\mathbb{D} : f(z) = 0\}$. Moreover, the sequence $(Q_n(1/f) \cdot f - 1)_{n\in\mathbb{N}}$ is uniformly bounded on $\overline{\mathbb{D}}$.

Known cases

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In all the previous cases, for each ζ in the unit **circle**, the set $\{Q_n(1/f)(\zeta) : n \in \mathbb{N}\}$ has only **one limit point**.

- **Question:** Is this always the case?

We will see that it is possible to find a (cyclic) function $f \in H^2$ such that $\{Q_n(1/f)(\zeta) : n \in \mathbb{N}\}$ **dense** in $\mathbb{C}$ for some ζ in $\partial\mathbb{D}$.

Theorem 1 (Bénéteau, Ivrii, M., Seco)
Let $E \subset \partial\mathbb{D}$ be a closed set of **arclength measure zero**. Then $\mathcal{U}_E :=$

$$\{f \in H^2 \setminus \{0\} : \forall g \in C(E)\ \exists (Q_{m_s}(1/f)) : Q_{m_s}(1/f) \to g \text{ in } C(E)\}$$

is a dense G_δ set in H^2. In particular, $\mathcal{U}_E \neq \emptyset$.

- **Remark:** The assumption on E cannot be dispensed with.

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▶ **Corollaries**

Let $E \subset \partial\mathbb{D}$ be a closed set of **arclength measure zero**.

1. There exists a function $f \in \mathcal{U}_E$ which is cyclic.
   In this case, $Q_n(1/f)(z)$ converges to $1/f(z)$ for all $z \in \mathbb{D}$, while $\{Q_n(1/f)(\zeta) : n \in \mathbb{N}\}$ is dense for all $\zeta \in E \subset \partial\mathbb{D}$!
2. Let (z_n) be a (finite or infinite) sequence in $\mathbb{D} \setminus \{0\}$ which satisfies the Blaschke condition

$$\sum_{n=1}^{\infty}(1 - |z_n|) < \infty.$$

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Proof of the Corollaries

Proposition

If g is an inner function in H^2 and $f \in H^2 \setminus \{0\}$, then, for each $n \in \mathbb{N}$,

$$Q_n(1/(g \cdot f)) = \overline{g(0)} \cdot Q_n(1/f).$$

Thus, if we additionally assume that $g(0) \neq 0$,

$$f \in \mathcal{U}_E \Leftrightarrow g \cdot f \in \mathcal{U}_E.$$

1. If E is as in Theorem 1 $\Rightarrow \exists\ F \in \mathcal{U}_E$. Since $F \in H^2$, we can write $F = F_I \cdot F_O$, where F_I is **inner** and F_O is **outer**. Hence $F_O \in \mathcal{U}_E$ and is cyclic (as an outer function).
2. We can obtain the function of Corollary 2 by multiplying the function $F_O \in \mathcal{U}_E$ which is cyclic (and so zero-free on $\mathbb{D}$) with a suitable Blaschke product (which is inner).

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What is behind the proof of Theorem 1?

Let $\{P_n : n \in \mathbb{N}\}$ be the set of polynomials with coefficients in $\mathbb{Q} + i\mathbb{Q}$ which do not vanish on E. For each $k, n, m \in \mathbb{N}$ we define:

$$E_{k,n,m} = \{f \in H^2 \setminus \{0\} : \|Q_m(1/f) - P_n\|_{C(E)} < 1/k\}.$$

- We observe that:

$$\mathcal{U}_E = \bigcap_{k,n=1}^{\infty} \bigcup_{m=1}^{\infty} E_{k,n,m}.$$

- In view of the **Baire category theorem**, it suffices to show:

Proposition (1)

For each $k, n, m \in \mathbb{N}$, we have that $E_{k,n,m}$ is **open** *in H^2.*

Proposition (2)

For each $k, n \in \mathbb{N}$, we have that $\bigcup_{m=1}^{\infty} E_{k,n,m}$ is **dense** *in H^2.*

What is behind the proof of Theorem 1?

Let $\{P_n : n \in \mathbb{N}\}$ be the set of polynomials with coefficients in $\mathbb{Q} + i\mathbb{Q}$ which do not vanish on E. For each $k, n, m \in \mathbb{N}$ we define:

$$E_{k,n,m} = \{f \in H^2 \setminus \{0\} : \|Q_m(1/f) - P_n\|_{C(E)} < 1/k\}.$$

- We observe that:

$$\mathcal{U}_E = \bigcap_{k,n=1}^{\infty} \bigcup_{m=1}^{\infty} E_{k,n,m}.$$

- In view of the **Baire category theorem**, it suffices to show:

Proposition (1)

For each $k, n, m \in \mathbb{N}$, we have that $E_{k,n,m}$ is **open** *in H^2.*

Proposition (2)

For each $k, n \in \mathbb{N}$, we have that $\bigcup_{m=1}^{\infty} E_{k,n,m}$ is **dense** *in H^2.*

What is behind Proposition (1) and (2)?

1. To show Proposition (1) we had to establish that, for each fixed n, the mapping $\mathcal{Q}_n : H^2 \setminus \{0\} \to C(E)$ with $\mathcal{Q}_n(f)$: the n^{th} o.p.a. to $1/f$ (restricted on E) is continuous.
2. To show Proposition (2) we had to prove a new result on **simultaneous zero-free approximation**.

• **Remark:** If we drop the 'zero-free' part, the corresponding result had been established by Beise and Müller, who used functional analysis techniques which could not be adapted to our case.

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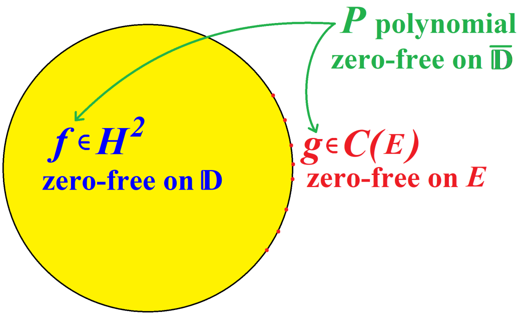

Theorem 2 (Bénéteau, Ivrii, M., Seco)

Let $E \subset \partial\mathbb{D}$ be a closed set of **arclength measure zero**. $\forall f \in H^2$ zero-free on $\mathbb{D}$ and $\forall g \in C(E)$ zero-free on E and $\forall \varepsilon > 0$, there is a polynomial P with **no zeros** on $\overline{\mathbb{D}}$ such that $\|f - P\|_{H^2} < \varepsilon$ and $\|g - P\|_{C(E)} < \varepsilon$.

Further directions

1. We recently established an analogue of Theorem 1 for the Dirichlet space D, by providing an analogous zero-free approximation result on $D \times C(E)$, where $E \subset \partial\mathbb{D}$ has **zero logarithmic capacity**.
2. Is it possible to obtain an analogue of Theorem 1 for the Bergman space A^2 on some sets E of positive arclength measure?
3. What is the behaviour of o.p.a on sets E that are not necessarily contained in the unit circle?

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Gracias!