Generalized Takagi Class

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January 30th, 2020

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The Takagi function

Let us consider the set $D = \left\{ \frac{k}{2^n} \in [0,1] : k, n \in \mathbb{Z}^+ \right\}$. For every $n \geq 0$, we define the set

$$D_n = \left\{ \frac{k}{2^n} \in [0,1] : k \in \mathbb{Z} \right\}.$$

The Takagi function $T : [0,1] \to \mathbb{R}$ is defined as

$$T(x) = \sum_{n=0}^{\infty} g_n(x)$$

where $g_n(x) = \text{dist}(x, D_n)$ denotes the distance from x to the set D_n.

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[1] T. Takagi, A simple example of the continuous function without derivative, Proc. Phys. Math. Soc. Tokio Ser. II (1) (1903) 176–177.

[2] P. C. Allaart, K. Kawamura, The improper infinite derivatives of Takagi's nowhere-differentiable function, J. Math. Anal. Appl. 372 (2) (2010) 656–665.

The Takagi function

Theorem (Góra and Stern, 2011)

If $x \in D$, then $\partial T(x) = \mathbb{R}$, otherwise $\partial T(x) = \emptyset$.

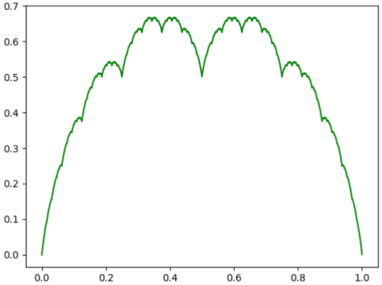

Figure: The Takagi function

[1] P. Góra and R. J. Stern, Subdifferential analysis of the Van der Waerden function, J. Convex Anal. 18 (3) (2011) 669–705.

The Takagi class

Let us consider the set $D = \left\{ \frac{k}{2^n} \in [0,1] : k, n \in \mathbb{Z}^+ \right\}$. For every $n \geq 0$, we define the set

$$D_n = \left\{ \frac{k}{2^n} \in [0,1] : k \in \mathbb{Z} \right\}.$$

The Takagi class is formed by all the functions $f_w : [0,1] \to \mathbb{R}$ defined by

$$f_w(x) = \sum_{n=0}^{\infty} w_n g_n(x)$$

where $w = (w_n)_n$ satisfies that $\sum_{n=0}^{\infty} |w_n| 2^{-n} < +\infty$.

Proposition (Hata and Yamaguti, 1984)

The Takagi class is a closed subspace of the space of continuous functions $C[0,1]$ isomorphic to ℓ_1.

[1] M. Hata and M. Yamaguti, The Takagi function and its generalization, Japan J. Appl. Math. 1 (1) (1984) 183–199.

Kôno's Differentiability Theorem

Theorem (Kôno, 1987)

1. If $w \in \ell_2$ then f_w is absolutely continuous and derivable almost everywhere.
2. If $w \in c_0$ but $w \notin \ell_2$ then f_w is non derivable almost everywhere, but it is derivable in an uncountable subset of $[0, 1]$, and the range of f'_w is $\mathbb{R}$.
3. If $w \notin c_0$ then f_w is nowhere derivable.

[1] N. Kôno, On generalized Takagi functions, Acta Math. Hungar. 49 (3–4) (1987) 315–324.

Takagi-Van der Waerden functions

Let $r \geq 2$ be an integer and $D = \left\{\frac{k}{r^n} \in [0,1] : k, n \in \mathbb{Z}^+\right\}$. We define

$$D_n = \left\{\frac{k}{r^n} \in [0,1] : k \in \mathbb{Z}\right\}.$$

The Takagi-Van der Waerden function $f_r : [0,1] \to \mathbb{R}$ is defined as

$$f_r(x) = \sum_{n=0}^{\infty} g_n(x)$$

The Takagi-Van der Waerden class

$$f_{r,w}(x) = \sum_{n=0}^{\infty} w_n g_n(x), \qquad \text{with } (w_n r^{-n})_n \in \ell_1$$

[1] B. L. van der Waerden, Ein einfaches Beispiel einer nicht-differenzierbaren stetigen Funktion, Math. Z. 32 (1) (1930) 474–475.

[2] F. A. Behrend, Some remarks on the construction of continuous non-differentiable functions, Proc. London Math. Soc. (2) 50 (1949) 463–481.

[3] J. Ferrera, J. Gómez Gil and J. Llorente, Infinite derivatives of the Takagi-Van der Waerden functions, J. Math. Anal. Appl. 479 (1) (2019) 987–1003.

Surveys on the Takagi function and its generalizations

Surveys

1. P. C. Allaart, K. Kawamura, The Takagi function: a survey, Real Anal. Exchange 37 (1) (2011/12) 1–54.
2. J. C. Lagarias, The Takagi function and its properties, in: Functions in number theory and their probabilistic aspects, RIMS Kôkyûroku Bessatsu, B34, Res. Inst. Math. Sci. (RIMS), Kyoto, 2012, pp. 153–189.

The Generalized Takagi class

Let D be a countable dense subset of $[0,1]$ such that $0,1 \in D$. Let $\mathcal{D} = (D_n)_n$ be a decomposition of D, that is an increasing sequence of finite sets D_n and a sequence $(\alpha_n)_n \in \ell_1$ satisfying:

1. $0,1 \in D_0$.
2. $D = \bigcup_n D_n$.
3. $|x-y| \leq \alpha_n$ for every $x,y \in D_n$ such that $(x,y) \cap D_n = \emptyset$.
4. There is $\rho \in (0,1]$ such that $|x-y| \geq \rho\alpha_n$ for every $x,y \in D_n$, $x \neq y$.

The Generalized Takagi Class is composed by all the functions $T_w : [0,1] \to \mathbb{R}$ defined by

$$T_w(x) = \sum_{n=0}^{\infty} w_n g_n(x)$$

where $w = (w_n)_n$ satisfies that $(w_n\alpha_n)_n \in \ell_1$.

[1] J. Ferrera, J. Gómez Gil, Generalized Takagi–Van der Waerden functions and their subdifferentials, J. Convex Anal. 25 (4) (2018) 1355–1369.

Notation and remarks

We denote by

1. $\mathcal{F}_n$ the family of connected components of $[0,1] \setminus D_n$.
2. $\widetilde{D}_n = \{\frac{a+b}{2} : a, b \in D_n, (a,b) \cap D_n = \emptyset\}$
3. $\widetilde{D} = \bigcup_n \widetilde{D}_n$.

Takagi-Van der Waerden Class

1. $D = \left\{\frac{k}{r^n} \in [0,1] : k, n \in \mathbb{Z}^+\right\}$.
2. $\mathcal{D} = (D_n)_n$ with $D_n = \{kr^{-n} \in [0,1] : k \in \mathbb{Z}\}$.
3. $\alpha_n = r^{-n}$ for every n and $\rho = 1$.

Special properties of the decomposition:

- $\widetilde{D}_n \subset D_{n+1}$ for every n (r even).
- $\widetilde{D}_n \subset \widetilde{D}_{n+1}$ for every n (r odd).

[1] J. Ferrera and J. Gómez-Gil, Differentiability of the functions of the Generalized Takagi Class, to appear in Rev. Mat. Complut., 2020

Generalization of Kôno's Theorem: The case $w \in c_0$

Theorem (Ferrera and Gómez Gil, 2020)

- If $w \in \ell_2$ and the series
$$\sum_{n \geq 1} \#(\widetilde{D}_{n-1} \setminus D_n)\alpha_n$$
converges, then T_w is absolutely continuous on $[0,1]$ and $T'_w(x) = \sum_n w_n g'_n(x)$ almost everywhere.
- If $w \notin \ell_2$ and the series
$$\sum_{n \geq 1} \#(D_n)\alpha_{n+1}$$
converges, then the set of points where T_w is derivable, denoted by $\mathcal{E}$, is a null set and $T'_w(\mathcal{E} \cap I) = \mathbb{R}$ for every interval $I \subset [0,1]$.

[1] J. Ferrera and J. Gómez-Gil, Differentiability of the functions of the Generalized Takagi Class, to appear in Rev. Mat. Complut., 2020

Generalization of Kôno's Theorem: The case $w \notin c_0$

Aim

To prove that under some restrictions on the decomposition, the functions of the Generalized Takagi Class are nowhere derivable provided that $w \notin c_0$.

Theorem (Ferrera, Gómez Gil and Llorente, 2019)

If the decomposition satisfies for every n that either

1. $\widetilde{D}_n \subset D_{n+1}$, or
2. $I \cap D_{n+1} \neq \emptyset$ for every $I \in \mathcal{F}_n$ and $\widetilde{D}_n \subset \widetilde{D}_{n+1}$,

then T_w is nowhere differentiable if and only if $w \notin c_0$.

[1] J. Ferrera, J. Gómez-Gil and J. Llorente, A characterization of the nowhere differentiable functions of the Generalized Takagi Class, Preprint 2019, `arXiv:1909.05545v1`

Non derivability on D

Proposition

Assume that the decomposition satisfies that for every n, either

1. $\widetilde{D}_n \subset D_{n+1}$, or
2. $I \cap D_{n+1} \neq \emptyset$ for every $I \in \mathcal{F}_n$ and $\widetilde{D}_n \subset \widetilde{D}_{n+1}$.

Then, T_w has not finite lateral derivatives at $x \in D$ provided that $w \notin c_0$.

Theorem (Ferrera, Gómez Gil and Llorente, 2019)

Let $w \notin c_0$. Assume that the decomposition satisfies that $D_{n+1} \cap I \neq \emptyset$ for every $I \in \mathcal{F}_n$, and that either $\rho > \frac{1}{2}$ or $\alpha_{n+1} \leq \rho\alpha_n$ for every n. If $x \in D$, then T_w has not finite lateral derivatives at x.

[1] J. Ferrera, J. Gómez-Gil and J. Llorente, A characterization of the nowhere differentiable functions of the Generalized Takagi Class, Preprint 2019, `arXiv:1909.05545v1`

Non derivability on D

Example

Let $w \notin c_0$ be defined as:

- $w_0 = 0,\ w_1 = 1,\ w_2 = -1,\ w_3 = -2^{-1}$.
- $w_{3k-2} = 1,\ w_{3k-1} = -1$ and $w_{3k} = 2^{-k}$ for every $k > 1$.

Let us consider a decomposition satisfying

- $D_{3k-2} = D_{3k-1}$, and
- $D_{3k} = D_{3(k+1)-2} = D_{3k-1} \cup \widetilde{D}_{3k-1}$ for every $k \geq 1$.

We have that $T'_w(x) = 0$ for every $x \in D_1$.

Theorem (Ferrera, Gómez Gil and Llorente, 2019)

Assume that $\alpha_{n+1} \leq \frac{\rho\alpha_n}{2}$ for every n. If $w \notin c_0$ and $x \in D$, then T_w is not derivable at x.

[1] J. Ferrera, J. Gómez-Gil and J. Llorente, A characterization of the nowhere differentiable functions of the Generalized Takagi Class, Preprint 2019, `arXiv:1909.05545v1`

Nowhere derivability

Example

Let us consider the sets $D_0^+ = \{1\}$ and $D_1^+ = \left\{\frac{2}{3}, 1\right\}$. For every integer $n \geq 1$ we define the sets

$$D_{2n}^+ = \left\{\frac{k}{2^n} \in (0,1] : k \in \mathbb{Z}\right\} \cup D_{2n-1}^+$$

and

$$D_{2n+1}^+ = \left\{\frac{k}{2^n} - \frac{1}{3^{n+1}} \in (0,1] : k \in \mathbb{Z}\right\} \cup D_{2n}^+.$$

For all $n \geq 0$ we also define $D_n^- = \{-x : x \in D_n^+\}$ and we consider the set $D_n = D_n^+ \cup D_n^-$. Let $w \notin c_0$ be defined as $w_{2n} = 1$ and $w_{2n+1} = -1$ for every n. Then, $T_w : [-1,1] \to \mathbb{R}$ is derivable at 0 and $T_w'(0) = 0$.

Nowhere derivability

Theorem (Ferrera, Gómez Gil and Llorente, 2019)

Assume that $w \notin c_0$. If the decomposition satisfies for every n that either

1. $\widetilde{D}_n \subset D_{n+1}$, or
2. $I \cap D_{n+1} \neq \emptyset$ for every $I \in \mathcal{F}_n$ and $\widetilde{D}_n \subset \widetilde{D}_{n+1}$,

then T_w is nowhere derivable.

Theorem (Ferrera, Gómez Gil and Llorente, 2019)

If $w \notin c_0$ and $w_k \geq 0$ for every k, then

- $\partial T_w(x) = \mathbb{R}$ provided that $x \in D$.
- $\partial T_w(x) = \emptyset$ provided that $x \notin D$.

[1] J. Ferrera, J. Gómez-Gil and J. Llorente, A characterization of the nowhere differentiable functions of the Generalized Takagi Class, Preprint 2019, arXiv:1909.05545v1

[2] P. Góra and R. J. Stern, Subdifferential analysis of the Van der Waerden function, J. Convex Anal. 18 (3) (2011) 669–705.

Acknowledgments

THANK YOU ALL!!!