Orthogonally additive polynomials on non-commutative L^p-spaces

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Introduction

Definition

Let X and Y be linear spaces. A map $P\colon X \to Y$ is said to be an *m-homogeneous polynomial* if there exists an m-linear map $\varphi : X^m \to Y$ such that

$$P(x) = \varphi(x, \ldots, x) \quad (x \in X).$$

Example

Let X be a linear space that has an additional structure that allow us to multiply its elements (algebra, function space, etc).
If $X_{(m)}$ is a linear space containing the set $\{x^m : x \in X\}$ and $\Phi : X_{(m)} \to Y$ is a linear map, then we can define an m-homogeneous polynomial $P : X \to Y$ as follows:

$$P(x) = \Phi(x^m) \quad (x \in X).$$

Question

If P is a polynomial on X, then $P(x) = \Phi(x^m)$ $(x \in X)$ for some linear map Φ?

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Answer: no.

Example

If $P(x) = \Phi(x^m)$ $(x \in X)$, then P verifies that

$$x, y \in X,\ xy = yx = 0 \implies P(x + y) = P(x) + P(y).$$

Let $P : \mathbb{M}_2 \to \mathbb{C}$, $P(A) = a_{11}a_{22}$ $(A = (a_{ij}) \in \mathbb{M}_2)$.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \implies P(A + B) \neq P(A) + P(B)$$

Let X and Y be linear spaces.

- We say that $x, y \in X$ are orthogonal if $xy = yx = 0$. In that case, we write $x \perp y$.
- A map $P\colon X \to Y$ is said to be orthogonally additive on a subset $\mathcal{S} \subset X$ if

$$x, y \in \mathcal{S},\ x \perp y \implies P(x+y) = P(x) + P(y).$$

- A map $P\colon X \to Y$ is said to be orthogonally additive if it is orthogonally additive on X.

Question

If P is a polynomial on X, and P is orthogonally additive on a certain subset $\mathcal{S} \subset X$, then $P(x) = \Phi(x^m)$ $(x \in X)$ for some linear map Φ?

Example

Let $P : C^1[0,1] \to \mathbb{C}$, $P(f) = f'(0)^2$ $(f \in C^1[0,1])$.
If $fg = 0$, then $f'(0)g'(0) = 0$, so P is orthogonally additive:

$$P(f+g) = f'(0)^2 + g'(0)^2 + 2f'(0)g'(0) = P(f) + P(g).$$

If $P(f) = \Phi(f^2)$ $(f \in C^1[0,1])$, then

$$P(f) = f'(0)^2 = (f+1)'(0)^2 = P(f+1) = \Phi(f^2 + 2f + 1)$$
$$= P(f) + 2\Phi(f) + P(1) \implies \Phi(f) = 0 \ (f \in C^1[0,1]).$$

Orthogonal additivity is not enough for polynomials on $C^1[0,1]$.

History of the problem

Theorem (Sundaresan (1991))

Let $1 \leq p < \infty$ and let $P : L^p[0,1] \to \mathbb{R}$ be an orthogonally additive continuous m-homogeneous polynomial. Then there exists a unique continuous linear map $\Phi : L^{p/m}[0,1] \to \mathbb{R}$ such that

$$P(f) = \Phi(f^m) \quad (f \in L^p[0,1]).$$

Theorem (Sundaresan (1991))

Let $1 \leq p < \infty$ and let $P : \ell^p \to \mathbb{R}$ be an orthogonally additive continuous m-homogeneous polynomial. Then there exists a unique continuous linear map $\Phi : \ell^{p/m} \to \mathbb{R}$ such that

$$P((x_n)_n) = \Phi((x_n^m)_n) \quad ((x_n)_n \in \ell^p).$$

Theorem (Pérez García, Villanueva (2005), Carando, Lasalle, Zalduendo (2006))

Let K be a compact topological space and Y be a Banach space. Let $P : C(K) \to Y$ be an orthogonally additive continuous m-homogeneous polynomial. Then there exists a continuous linear map $\Phi : C(K) \to Y$ such that

$$P(f) = \Phi(f^m) \quad (f \in C(K)).$$

Theorem (Palazuelos, Peralta, Villanueva (2008))

Let A be a C^*-algebra and X be a Banach space. Let $P : A \to X$ be a continuous m-homogeneous polynomial. Then, the following are equivalent:

1. *P is orthogonally additive;*
2. *P is orthogonally additive on A_{sa};*
3. *there exists a continuous linear map $\Phi : A \to X$ such that $P(x) = \Phi(x^m)$ $(x \in A)$.*

Theorem (Villena (2017))

Let X be a Banach space and $n \in \mathbb{N}$. Let $P : C^n([0,1]) \to X$ be a continuous m-homogeneous polynomial. If P is orthogonally additive, then for each $(n_1, \dots, n_m) \in \mathbb{Z}^m$ with $0 \le n_1 \le \cdots \le n_m \le n$ there exists a continuous linear map $T_{(n_1,\dots,n_m)} : C^{n-n_m}([0,1]) \to X$ such that

$$P(f) = \sum_{0 \le n_1 \le \cdots \le n_m \le n} T_{(n_1,\dots,n_m)}\big(f^{(n_1)} \cdots f^{(n_m)}\big)$$

for each $f \in C^n([0,1])$.

Notation

Let X be a Banach space.

- $\mathcal{F}(X)$ = finite-rank operators on X.
- $\mathcal{A}(X) = \overline{\mathcal{F}(X)}$, approximable operators on X.

Theorem (Alaminos, Godoy, Villena (2019))

Let X and Y be Banach spaces and suppose that X^* has the BAP. Let $P : \mathcal{A}(X) \to Y$ be a continuous m-homogeneous polynomial. Then, the following are equivalent:

1. *P is orthogonally additive;*
2. *P is orthogonally additive on $\mathcal{F}(X)$;*
3. *there exists a continuous linear map $\Phi : \mathcal{A}(X) \to Y$ such that $P(T) = \Phi(T^m)$ $(T \in \mathcal{A}(X))$.*

Non-commutative L^p-spaces

Definition

Let $\mathcal{M}$ be a von Neumann algebra. A *trace* on $\mathcal{M}$ is a map $\tau : \mathcal{M}_+ \to [0, \infty]$ satisfying:

- $\tau(x + y) = \tau(x) + \tau(y)$ for all $x, y \in \mathcal{M}_+$.
- $\tau(\lambda x) = \lambda \tau(x)$ for all $x \in \mathcal{M}_+$ and $\lambda \geq 0$.
- $\tau(xx^*) = \tau(x^*x)$ for all $x \in \mathcal{M}$.

1. τ is *normal* is $\sup_\alpha \tau(x_\alpha) = \tau(\sup_\alpha x_\alpha)$ for any bounded increasing net (x_α) in $\mathcal{M}_+$.
2. τ is *semifinite* if for any non-zero $x \in \mathcal{M}_+$ there is a non-zero $y \in \mathcal{M}_+$ such that $y \leq x$ and $\tau(y) < \infty$.
3. τ is *faithful* if $\tau(x) = 0$ implies $x = 0$.

$\mathcal{M}$ is said to be *semifinite* if it admits a normal semifinite faithful trace.

- Let $\mathcal{M}$ be a semifinite von Neumann algebra with normal semifinite faithful trace τ.
- Let $S_+(\mathcal{M},\tau) = \{x \in \mathcal{M}_+ : \tau(\text{supp}(x)) < \infty\}$ and $S(\mathcal{M},\tau) = \text{lin} S_+(\mathcal{M},\tau)$.
- If $0 < p < \infty$ we define

$$\|x\|_p = \big(\tau(|x|^p)\big)^{1/p}, \quad (x \in S).$$

  $\|\cdot\|_p$ is a norm if $p \geq 1$ and it is a p-norm if $p < 1$.
- $L^p(\mathcal{M},\tau)$ is the completion of $(S(\mathcal{M},\tau), \|\cdot\|_p)$.
- We set $L^\infty(\mathcal{M},\tau) = (\mathcal{M}, \|\cdot\|)$ and $L^0(\mathcal{M},\tau) =$ measurable closed densely defined operators affiliated to $\mathcal{M}$.
- $L^p(\mathcal{M},\tau) = \{x \in L^0(\mathcal{M},\tau) : \big(\tau(|x|^p)\big)^{1/p} < \infty\}$.

Properties

- $L^p(\mathcal{M}, \tau)$ is a Banach space if $1 \leq p \leq \infty$.
- $L^p(\mathcal{M}, \tau)$ is a quasi-Banach space if $0 < p < 1$.
- Hölder's inequality: if $0 < p, q, r \leq \infty$ are such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, then

$$x \in L^p(\mathcal{M}, \tau),\ y \in L^q(\mathcal{M}, \tau) \implies xy \in L^r(\mathcal{M}, \tau)$$

and $\|xy\|_r \leq \|x\|_p \|y\|_q$.

Definition

We say that $x, y \in L^0(\mathcal{M}, \tau)$ are *mutually orthogonal* if $xy^* = y^*x = 0$. We write $x \perp y$.
Let X be a linear space. A map $P : L^p(\mathcal{M}, \tau) \to X$ is said to be *orthogonally additive* on a subset $\mathcal{S}$ of $L^p(\mathcal{M}, \tau)$ if

$$x, y \in \mathcal{S},\ x \perp y \implies P(x + y) = P(x) + P(y).$$

Example

Let H be a Hilbert space and let Tr be the usual trace on the von Neumann algebra $\mathcal{B}(H)$. Then $L^p(\mathcal{B}(H), \mathrm{Tr})$ is the Schatten class $S^p(H)$.

Properties:

- $\mathcal{F}(H) \subset S^p(H) \subset \mathcal{K}(H)$.
- If $0 < p < q$, we have $S^p(H) \subset S^q(H)$ and $\|x\| \leq \|x\|_q \leq \|x\|_p$ $(x \in S^p(H))$.
- $S(\mathcal{B}(H), \mathrm{Tr}) = \mathcal{F}(H)$.

Main result

Theorem

Let $\mathcal{M}$ be a von Neumann algebra with a normal semifinite faithful trace τ, let X be a topological linear space, and let $P\colon L^p(\mathcal{M},\tau) \to X$ be a continuous m-homogeneous polynomial with $0 < p < \infty$. Then the following conditions are equivalent:

1. *there exists a continuous linear map $\Phi\colon L^{p/m}(\mathcal{M},\tau) \to X$ such that $P(x) = \Phi(x^m)$ $(x \in L^p(\mathcal{M},\tau))$;*
2. *P is orthogonally additive on $L^p(\mathcal{M},\tau)_{\mathrm{sa}}$;*
3. *P is orthogonally additive on $S(\mathcal{M},\tau)_+$.*

If the conditions are satisfied, then the map Φ is unique.

Proposition

Let H be a Hilbert space with $\dim H \geq 2$, let X be a topological linear space, and let $P\colon S^p(H) \to X$ be a continuous m-homogeneous polynomial. Suppose that P is orthogonally additive on $S^p(H)$. Then $P = 0$.

Proof (sketch):

- $P\,|_{\mathcal{F}(H)_{sa}} = 0 \implies P\,|_{\mathcal{F}(H)} = 0 \implies P = 0$.
- For each $\xi, \eta \in H$, let $\xi \otimes \eta \in \mathcal{F}(H)$ defined by $\big(\xi \otimes \eta\big)(\psi) = \langle \psi | \eta \rangle \xi$ $(\psi \in H)$.
- If $x \in \mathcal{F}(H)_{\mathrm{sa}}$, then $x = \sum_{j=1}^{k} \alpha_j \xi_j \otimes \xi_j$, where $k \geq 2$, $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$, and $\{\xi_1, \ldots, \xi_k\} \subset H$ is orthonormal.
- If $\mathcal{M} = \operatorname{alg}\{\xi_i \otimes \xi_j : i, j \in \{1, \ldots, k\}\} \subset \mathcal{F}(H)$, then $\mathcal{M}$ is $*$-isomorphic to the von Neumann algebra $\mathcal{B}(K)$, where $K = \operatorname{lin}\{\xi_1, \ldots, \xi_k\}$.
- $P\,|_{\mathcal{M}} = 0 \implies P(x) = 0$.

Lemma

Let $\mathcal{M}$ be a von Neumann algebra, let X be a topological linear space, and let $P\colon \mathcal{M} \to X$ be a continuous m-homogeneous polynomial. Then the following conditions are equivalent:

1. *there exists a continuous linear map $\Phi\colon \mathcal{M} \to X$ such that $P(x) = \Phi(x^m)$ $(x \in \mathcal{M})$;*
2. *P is orthogonally additive on $\mathcal{M}_{\mathrm{sa}}$;*
3. *P is orthogonally additive on $\mathcal{M}_+$.*

If the conditions are satisfied, then the map Φ is unique.

Proof (sketch):

- Let $\varphi : \mathcal{M}^m \to X$ be the symmetric m-linear map associated with P and define $\Phi : \mathcal{M} \to X$ by

$$\Phi(x) = \varphi(x, 1, \ldots, 1) \quad (x \in \mathcal{M}).$$

- Let $Q : \mathcal{M} \to X$ be defined by $Q(x) = \Phi(x^m)$ $(x \in \mathcal{M})$.
- If $P\mid_{\mathcal{M}_{sa}} = Q\mid_{\mathcal{M}_{sa}}$, then $P = Q$.
- Let $\{e_1, \ldots, e_k\} \in \mathcal{M}$ be mutually orthogonal projections, let $\{\rho_1, \ldots, \rho_k\} \subset \mathbb{R}$ and let $x = \sum_{j=1}^k \rho_j e_j$.
- $P(x) = \sum_{j=1}^k \rho_j^m \varphi(e_j, \ldots, e_j) = Q(x)$.
- If $x \in \mathcal{M}_{sa}$ there exists $(x_n) \subset \mathcal{M}_{sa}$ such that $\lim x_n = x$ and x_n has finite spectrum.
- $P(x) = \lim P(x_n) = \lim Q(x_n) = Q(x)$.

Proof of the theorem (sketch):

- Let $e \in \text{Proj}(M)$ with $\tau(e) < \infty$ and let $\mathcal{M}_e = e\mathcal{M}e$.
- $\mathcal{M}_e \subset S(\mathcal{M}, \tau)$.
- $P\mid_{\mathcal{M}_e}$ is continuous.
- There exists a unique continuous linear map $\Phi_e : \mathcal{M}_e \to X$ such that $P(x) = \Phi_e(x^m)$ $(x \in \mathcal{M}_e)$.
- For each $x \in S(\mathcal{M}, \tau)$, define $\Phi(x) = \Phi_e(x)$, where $e \in \text{Proj}(\mathcal{M})$ is such that $\tau(e) < \infty$ and $x \in \mathcal{M}_e$.
- Φ is linear.
- Φ is continuous with respect to the norm $\|\cdot\|_{p/m}$.
- Φ extends to a continuous linear map from $L^{p/m}(\mathcal{M}, \tau)$ to the completion of X.
- $\Phi(L^{p/m}(\mathcal{M}, \tau)) \subset X$.

Proposition

Let $\mathcal{M}$ be a von Neumann algebra with a normal semifinite faithful trace τ and with no minimal projections, let X be a Banach space, and let $\Phi\colon L^p(\mathcal{M},\tau) \to X$ be a continuous linear map with $0 < p < 1$. Then $\Phi = 0$.

Proof (sketch):

- For each projection $e \in \mathcal{M}$ with $\tau(e) < \infty$ and each $0 \le \rho \le \tau(e)$, there exists a projection $e_0 \in \mathcal{M}$ such that $e_0 \le e$ and $\tau(e_0) = \rho$.
- Let $e_0 \in \mathrm{Proj}(M)$ with $\tau(e_0) < \infty$.
- There is a decreasing sequence of projections (e_n) such that $\tau(e_n) = 2^{-n}\tau(e_0)$ and $\|\Phi(e_0)\| \le 2^n\|\Phi(e_n)\| \quad (n \in \mathbb{N})$.
- $\left\|2^n e_n\right\|_p \to 0 \implies 2^n\Phi(e_n) \to 0 \implies \Phi(e_0) = 0$.
- Take $x \in S(\mathcal{M},\tau)_+$, and let $e = \mathrm{supp}(x)$. There exists $(x_n) \subset \mathcal{M}_e$ such that $\lim x_n = x$ and $\Phi(x_n) = 0$.

Proposition

Let $\mathcal{M}$ be a von Neumann algebra with a normal semifinite faithful trace τ and with no minimal projections, let X be a Banach space, and let $P\colon L^p(\mathcal{M},\tau) \to X$ be a continuous m-homogeneous polynomial with $0 < p/m < 1$. Suppose that P is orthogonally additive on $S(\mathcal{M},\tau)_+$. Then $P = 0$.

Example

Suppose that $0 < p/m < 1$. Let $\Phi : L^{p/m}(\mathcal{M},\tau) \to L^{p/m}(\mathcal{M},\tau)$,
$\Phi(x) = x, \ (x \in L^{p/m}(\mathcal{M},\tau))$.
The polynomial $P : L^p(\mathcal{M},\tau) \to L^{p/m}(\mathcal{M},\tau)$,

$$P(x) = \Phi(x^m) = x^m \quad (x \in L^p(\mathcal{M},\tau))$$

is orthogonally additive on $S(\mathcal{M},\tau)_+$.

Corollary

Let $\mathcal{M}$ be a von Neumann algebra with a normal semifinite faithful trace τ, and let $P\colon L^p(\mathcal{M},\tau) \to \mathbb{C}$ be a continuous m-homogeneous polynomial with $m \leq p < \infty$. Then the following conditions are equivalent:

1. *there exists $\zeta \in L^r(\mathcal{M},\tau)$ such that $P(x) = \tau(\zeta x^m)$ $(x \in L^p(\mathcal{M},\tau))$, where $r = p/(p-m)$ (with the convention that $p/0 = \infty$);*
2. *P is orthogonally additive on $L^p(\mathcal{M},\tau)_{\mathrm{sa}}$;*
3. *P is orthogonally additive on $S(\mathcal{M},\tau)_+$.*

If the conditions are satisfied, then ζ is unique and $\|P\| \leq \|\zeta\|_r \leq 2\|P\|$; moreover, if P is hermitian, then ζ is self-adjoint and $\|\zeta\|_r = \|P\|$.

References

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