Weierstrass M test: algebraic genericity

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M-Weierstrass' Theorem

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If there exists $(c_n)_n \subset \mathbb{R}$ such that $|f_n(x)| \leq c_n$ for all $x \in [0,1]$, $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} c_n < +\infty$, then the series is uniformly convergent on $[0,1]$.

Counterexample

Consider the series $\sum_{n=1}^{\infty} f_n(x)$ where $f_n \in \mathcal{C}([0,1])$ is given by

$$f_n(x) = \begin{cases} \frac{1}{n}\sin^2\left(2^{n+1}\pi x\right) & \text{if } x \in \left(\frac{1}{2^{n+1}}, \frac{1}{2^n}\right) \\ 0 & \text{otherwise.} \end{cases}$$

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We have that

- The series is absolutely convergent.
- The series is uniformly convergent on $[0, 1]$.
- The series has not a mayorant sequence.

Definition of Anti-M Weierstrass' Sequence

The family $\mathcal{A} \subset c_0(C(I))$ of anti-M Weierstrass sequences of functions, where an element $f = (f_n)_n \in \mathcal{A}$ must fulfill the following conditions:

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(A3) The series $\sum_{n=1}^{\infty} f_n(x)$ does not possess a mayorant, that is

$$\sum_{n=1}^{\infty} \|f_n\|_{\infty} = +\infty$$

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- If in addition, $\mathcal{A}$ is a commutative algebra, we say that $\mathcal{B}$ is strongly algebrable if $\mathcal{B} \cup \{0\}$ contains generated algebra which is isomorphic to a free algebra.

Previous concepts

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- there exists

$$\liminf_{n\to\infty} \|u_n\|_\infty =: L > 0. \tag{F3}$$

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1. *The series $\sum_{n=1}^{\infty} a_n u_n(x)$ converges absolutely on I.*
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3. *The series* $\sum_{n=1}^{\infty} \|a_n u_n\|_\infty < +\infty$ *if and only if* $a \in l_1$*.*

Examples

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Let $I = [a, b] \subset \mathbb{R}$ *and consider the sequence* $u = (u_n)_n \in C(I)^{\mathbb{N}}$ *given by*

$$u_n(x) = \begin{cases} \sin\left(2^n\pi\left(\frac{x-a}{b-a}\right) - \pi\right) & \text{if } x \in I_n, \\ 0 & \text{otherwise.} \end{cases}$$

where $I_n = \left[\frac{(2^n-1)a+b}{2^n}, \frac{(2^{n-1}-1)a+b}{2^{n-1}}\right]$.

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If $a = (a_n)_n \in c_0 \setminus l_1$ *then*

$$f = (a_n u_n)_n \in \mathcal{A}.$$

Turning a continuous function into an Anti-M sequence

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- *Define*

$$u_n(x) := f\left(2^n\left(\frac{x-a}{b-a}\right) - 1\right), \quad \forall x \in I_{3n-1}.$$

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- *Choosing any* $a = (a_n)_n \in c_0 \setminus l_1$*, we obtain that*

$$f = (a_n u_n)_n \in \mathcal{A}.$$

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Theorem

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- Let $H \subset (0, +\infty)$ be a $\mathbb{Q}$-linearly independent set, $\operatorname{card}(H) = \mathfrak{c}$. Consider

$$f_{n,c}(x) := a_{n,c} u_n(x),$$

where $a_{n,c}$ is chosen as $a_{n,c} = \dfrac{1}{\log^c(n)}$.

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- Let $\mathcal{B}$ be the algebra generated by $\{(f_{n,c})_n : c \in H\}$.

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Theorem (Bartoszewicz, Glab)

The set $c_0 \setminus \bigcup_{p \geq 1} l_p$ *is densely strongly* $\mathfrak{c}$*-algebrable in* c_0.

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Theorem

Let $\mathfrak{A}$ be a free algebra as above and $u = (u_n)_n \in \mathcal{F}$. Then, the algebra generated by the set

$$\{(a_n u_n(x))_n \ : \ a = (a_n)_n \in \mathfrak{A}\}$$

is free in the family of anti-M sequences of functions $\mathcal{A}$

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Corollary

The family $\mathcal{A}$ of anti-M Weierstrass sequences is maximal lineable.

Dense-lineability

Lemma (Aron, García, Pérez, Seoane)

Let X be a separable metrizable topological vector space, $A \subset X$ maximal lineable and $B \subset X$ dense-lineable in X with $A \cap B = \varnothing$. If A is stronger than B then A is maximal dense-lineable.

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- $c_{00}(\mathcal{C}(I))$ is a dense-lineable subset of $c_0(\mathcal{C}(I))$.
- $c_{00}(\mathcal{C}(I))) + \mathcal{A} \subseteq \mathcal{A}$.

Thank you very much for your attention

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