C^1- fine approximations without critical points

Miguel García Bravo

(Universidad Autónoma de Madrid/ICMAT)

V Congreso de Jóvenes Investigadores de la RSME, Castellón
27-January-2020

Critical point

Definition (Critical point)

If we have a function $f : E \longrightarrow F$ between Banach spaces which is (Fréchet) differentiable at some point x we will say that x is a critical point if $Df(x) \in L(E, F)$ is not a surjective operator.

- Set of critical points: C_f
- Set of critical values: $f(C_f)$

Recall that the (Fréchet) derivative $Df(x)$ of f at x is defined as the unique linear continuous operator such that

$$\lim_{h \to 0} \frac{||f(x+h) - f(x) - Df(x)(h)||}{||h||} = 0.$$

QUESTION: Which regularity conditions do we have to impose to f so that $f(C_f)$ is *small* in some sense?

Classical Morse-Sard theorem: $\dim(E), \dim(F) < \infty$

Theorem (Morse 1939, Sard 1942)

Let $f : \mathbb{R}^n \to \mathbb{R}^d$ be a C^k function with $k \geq \max\{n-d+1, 1\}$. Then the set of critical values, $f(C_f)$, is of Lebesgue measure zero in $\mathbb{R}^d$ ($\mathcal{L}^d(f(C_f)) = 0$).

Note that here

$$C_f = \{x \in \mathbb{R}^n : rank\, Df(x) < \min\{n, d\}\}.$$

This result has been shown to be sharp in the class of functions C^j thanks to the famous counterexample of Whitney in 1935. He built a function $f : \mathbb{R}^2 \to \mathbb{R}$ of class C^1 such that $\mathcal{L}^1(f(C_f)) > 0$.

This theorem has been generalized to other function spaces such as Hölder spaces $C^{k-1,1}$, Sobolev spaces $W^{k,p}$ with $p > n$ or to the space of functions of bounded variation BV_n.

Infinite dimension: $\dim(E) = \infty$

The first version of the Morse-Sard Theorem in infinite dimensions:

Theorem (Smale 1965)

If E and F are separable Banach spaces and $f : E \to F$ is a C^r Fredholm mapping (that is, every differential $Df(x)$ is a Fredholm operator between E and F) then $f(C_f)$ is meager, and in particular $f(C_f)$ has no interior points, provided that $r > \max\{index(Df(x)), 0\}$ for all $x \in E$.

However these assumptions are quite restrictive as, for instance, if E is infinite-dimensional then no linear bounded operator $T : E \to \mathbb{R}$ is Fredholm.

Kupka's counterexample (1965): there are C^∞ functions $f : \ell_2 \to \mathbb{R}$ so that their sets of critical values $f(C_f)$ contain intervals.

Conclusion: We cannot expect to have a good version of the Morse-Sard theorem for infinite dimensions !!

Example (Bates-Moreira's counterexample, 2001)

Define $f : \ell_2 \to \mathbb{R}$,

$$f(\sum_{n=1}^{\infty} x_n e_n) = \sum_{n=1}^{\infty} (3 \cdot 2^{-\frac{n}{3}} x_n^2 - 2x_n^3),$$

which is a polynomial of degree three and whose set of critical points is $C_f = \{\sum_{n=1}^{\infty} x_n e_n : x_n \in \{0, 2^{-\frac{n}{3}}\}\}$ and $f(C_f) = [0, 1]$.

Approximated Morse-Sard results

Given a pair of Banach spaces (E, F) we say that the property (P_k) holds if for every continuous mapping $f : E \to F$ and every continuous function $\varepsilon : E \to (0, \infty)$ there exists a C^k mapping $g : E \to F$ such that

1. $\|f(x) - g(x)\| \leq \varepsilon(x)$ for every $x \in E$, and
2. $Dg(x) : E \to F$ is a surjective linear operator for every $x \in E$.

Theorem

- **(Azagra, Cepedello 2004):** *Let E be a separable Hilbert space. Then property (P_∞) holds for the pair $(E, \mathbb{R}^d)$.*
- **(Azagra, Jiménez-Sevilla 2007):** *Let E be a Banach space with separable dual. Then property (P_1) holds for the pair $(E, \mathbb{R})$.*
- **(Azagra, Dobrowolski, García-Bravo 2019):** *Let $E = c_0, \ell_p, L^p$, $1 < p < \infty$. Let F be a Banach space, and assume that there exists a bounded linear surjective operator from E onto F. Then property (P_k) holds for the pair (E, F), where k denotes the order of smoothness of the norm of the space E.*

Different question

We are given a continuous function $f : E \to \mathbb{R}^d$ of class C^1 and so that C_f is included in some open set U.

Question: Are we able not only to uniformly approximate f by another C^1 function g without critical points but also

- to make g be equal to f outside U?
- to get an approximation in the C^1-fine topology?

Theorem (M. García-Bravo 2019)

Let $E = c_0, \ell_p$ with $1 < p < \infty$. Let $f : E \to \mathbb{R}^d$ be a C^1 function and $\varepsilon : E \to (0, \infty)$ a continuous function. Take any open set $U \supset C_f$. Then there exists a C^1 function $g : E \to \mathbb{R}^d$ such that,

1. $||f(x) - g(x)|| \leq \varepsilon(x)$ *for all* $x \in E$*;*
2. $f(x) = g(x)$ *for all* $x \in E \setminus U$*;*
3. $Dg(x)$ *is surjective for all* $x \in E$*, i.e.* φ *has no critical points; and*
4. *If* $E = c_0$ *we also get* $||Df(x) - Dg(x)|| \leq \varepsilon(x)$ *for all* $x \in E$*.*

Sketch of the proof: $f: c_0 \to \mathbb{R}$, $\varepsilon(x) = \varepsilon > 0$

Theorem (Real-valued case of c_0)

Let $f : c_0 \to \mathbb{R}$ be a C^1 function so that $C_f = \{x \in c_o : f'(x) = 0\} \subset U$ for some open set. Then for every $\varepsilon > 0$ there exists a C^1 function $g : c_0 \to \mathbb{R}$ such that,

1. $|f(x) - g(x)|, ||f'(x) - g'(x)|| \leq \varepsilon$ *for all* $x \in c_0$;
2. $f(x) = g(x)$ *for all* $x \in c_0 \setminus U$; *and*
3. $g'(x) \neq 0$ *for all* $x \in c_0$, *i.e.* g *has no critical points;*

- **Step 1**: Construct a C^1 function $g : U \to \mathbb{R}$ such that
  1. $|f(x) - g(x)| \leq \varepsilon$ for every $x \in U$.
  2. $||f'(x) - g'(x)|| \leq \varepsilon$ for every $x \in U$.
  3. $C_g = \emptyset$

  Use of C^1-fine approximation methods due to N. Moulis (1971).
- **Step 2**: We extend the function g to the whole space c_0 by letting it be equal to f outside U. Because of the C^1-fine approximation of Step 1 this extension is still of class C^1 on c_0 and also $C_g = \emptyset$

Ideas behind the proof

IMPORTANT FACT: c_0 has an equivalent norm of class C^∞ which locally depends on finitely many coordinates (LFC). We will work with this norm.

Definition

$||\cdot||$ is LFC if for every $x \in c_0$ there exists $m_x \in \mathbb{N}$, an open set U_x, some linear functionals $L_1, \ldots, L_{m_x} \in c_0^* = \ell_1$ and a function $\gamma : \mathbb{R}^{m_x} \to \mathbb{R}$ so that for all $y \in U_x$,

$$||y|| = \gamma(L_1(y), \ldots L_{m_x}(y)).$$

In particular, if $||\cdot||$ is smooth, $D||\cdot||(y) \in span\{L_1, \ldots, L_{m_x}\}$.

Note: $||\cdot||_\infty$ on c_0 is LFC but is not of class C^1.

Ideas behind the proof

Firstly define $\{h_j\}_{j\in\mathbb{N}}$ to be a C^∞ partition of unity subordinate to some open covering $\{B(x^j, r_j)\}_{j\in\mathbb{N}}$. By the property on the norm of being LFC for each ball $B(x^j, r_j)$ there is $L_{j(1)}, \dots, L_{j(m_j)} \in \ell_1$ so that

$$D||\cdot||(y) \in span\{L_{j(1)}, \dots, L_{j(m_j)}\} \ \forall y \in B(x^j, r_j).$$

Therefore for all $y \in U$ and $j \in \mathbb{N}$
$D||\cdot||(y) \in span\{L_{k(1)}, \dots, L_{k(m_k)} : k \in \mathbb{N}\}$ and also
$h'_j(y) \in span\{L_{k(1)}, \dots, L_{k(m_k)} : k \in \mathbb{N}\}$.
Secondly use a technique from Moulis (1971), used for C^1-fine approximations, to find for each ball $B(x^j, r_j)$ an approximation of $f(x^j) + f'(x^j)(x - x^j) - f(x)$ by a C^1 function δ_j so that

- $||\delta'_j(y)||$ is small.
- $\delta'_j(y) \in span\{L_{k(1)}, \dots, L_{k(m_k)}, e^*_k : k \in \mathbb{N}\}$.

Let us define finally

$$g(x) := \sum_{j=1}^{\infty} h_j(x) \left[f(x^j) + f'(x^j)(x - x^j) - \delta_j(x) + T_j(x - x^j) \right].$$

Ideas behind the proof

Definition of T_j: It is an element of $L(c_0, \mathbb{R}) = c_0^* = \ell_1$ satisfying that

$$\begin{cases} ||T_j|| \text{ is small.} \\ T_j \notin span\{e_k^*, f'(x^k), L_{k(1)}, \dots, L_{k(l_k)}, T_1, \dots, T_{j-1}, : k \in \mathbb{N}\}. \end{cases}$$

We have

$$g(x) = \sum_{j=1}^{\infty} h_j(x) \left[f(x^j) + f'(x^j)(x - x^j) - \delta_j(x) + T_j(x - x^j) \right]$$

$$g'(y) = \sum_{j=1}^{n} h_j'(y) \left[f(x^j) + f'(x^j)(y - x^j) - \delta_j(y) + T_j(y - x^j) \right] + h_j(y) \left[f'(x^j) - \delta_j'(y) + T_j \right], \quad \text{for all } y \in B(x^n, r_n).$$

One can check that $|f(x) - g(x)|, ||f'(x) - g'(x)|| \leq \varepsilon$ for all $x \in U$.

Ideas behind the proof

For all $y \in B(x^n, r_n)$ we have

$$g'(y) = \sum_{j=1}^{n} h_j'(y) \left[f(x^j) + f'(x^j)(y - x^j) - \delta_j(y) + T_j(y - x^j)\right] + h_j(y) \left[f'(x^j) - \delta_j'(y) + T_j\right]$$

Observations:

- $h_j'(y) \in span\{L_{k(1)}, \ldots, L_{k(m_k)} : k \in \mathbb{N}\}$ for all $y \in U$ and $j = 1, \ldots, n$.
- $\delta_j'(y) \in span\{L_{k(1)}, \ldots, L_{k(m_k)}, e_k^* : k \in \mathbb{N}\}$ for all $y \in U$ and $j = 1, \ldots, n$.

However
$T_j \notin span\{e_k^*, f'(x^k), L_{k(1)}, \ldots, L_{k(m_k)}, T_1, \ldots, T_{j-1}, : k \in \mathbb{N}\}$, hence

$$T_n \notin span\{h_j'(y), f'(x^j), \delta_j'(y), T_1, \ldots, T_{n-1} : j = 1, \ldots, n\}$$

and so we can conclude that $g'(y) \neq 0$ for all $y \in U$.

Open question: C^k-fine approximations

An important open question in the theory of smooth approximations in Banach spaces is the following.

Problem

If we have a C^k function $f : E \to F$ between Banach spaces. Can we find a C^p $(p > k)$ function $g : E \to F$ that approximates f in the C^k-fine topology.

What is known:

1. Every C^k function $f : \mathbb{R}^n \to \mathbb{R}^d$ can be uniformly approximated by analytic ones in the C^k-fine topology (Whiteny, 1934).
2. For $E = c_0, \ell_p$, $1 < p < \infty$, every C^1 function $f : E \to F$ can be approximated by C^k functions in the C^1-fine topology (Moulis 1971), with k being the order of smoothness of the space E.
3. In the space c_0 **we cannot** approximate C^2 functions by C^∞ functions in the C^2-fine topology (Wells, 1969).

THANK YOU FOR YOUR ATTENTION !!