Some open problems on sequential properties of dual Banach spaces

V Congreso de Jóvenes Investigadores de la RSME

Análisis Funcional

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- *K is said to have* **countable tightness** *if for every subspace F of K, every point in the closure of F is in the closure of a countable subspace of F.*

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$$\Downarrow$$

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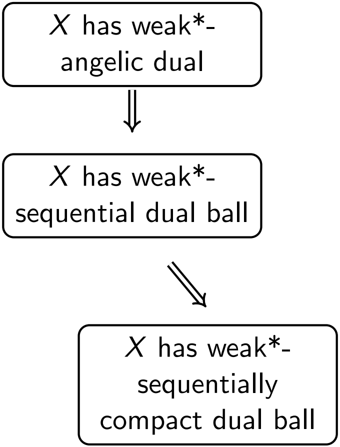
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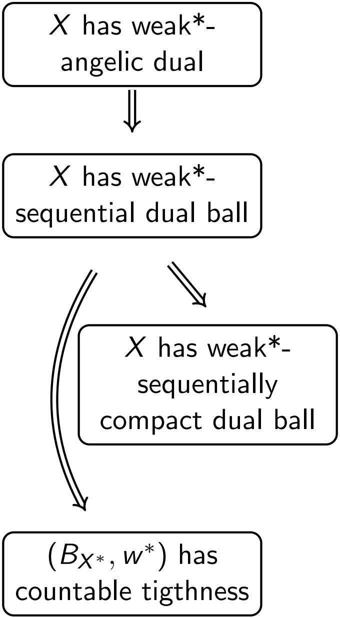
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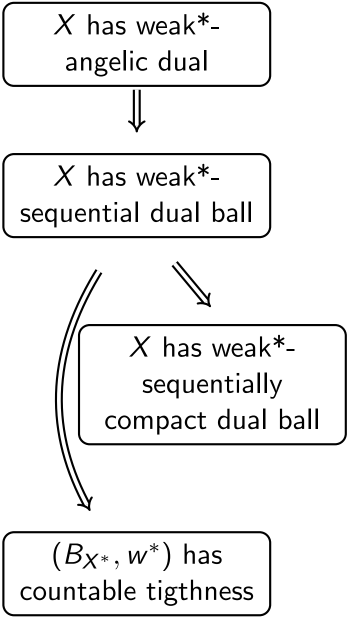
- *X has* **property (C)** *of Corson if and only if every point in the closure of C is in the weak*-closure of a countable subset of C for every convex set C in B_{X^*} (Pol's characterization).*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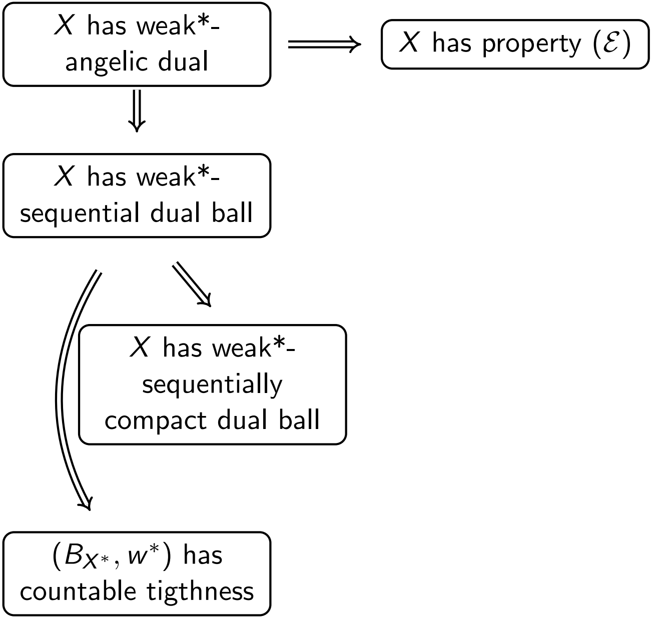
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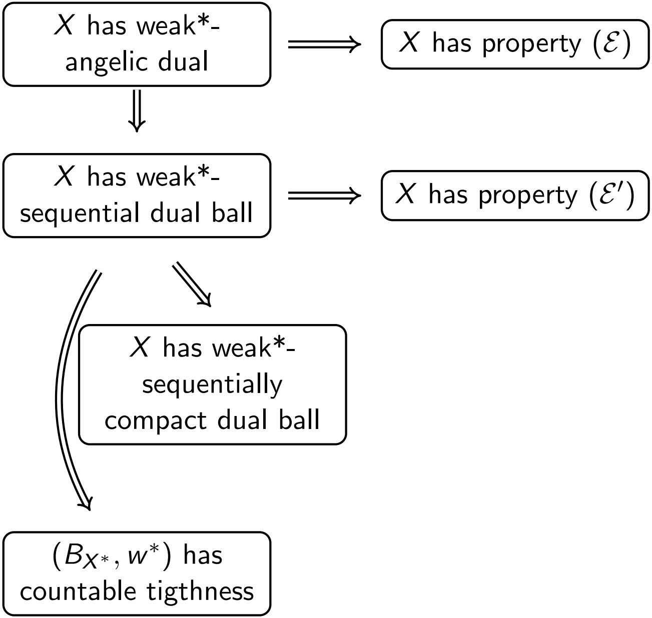

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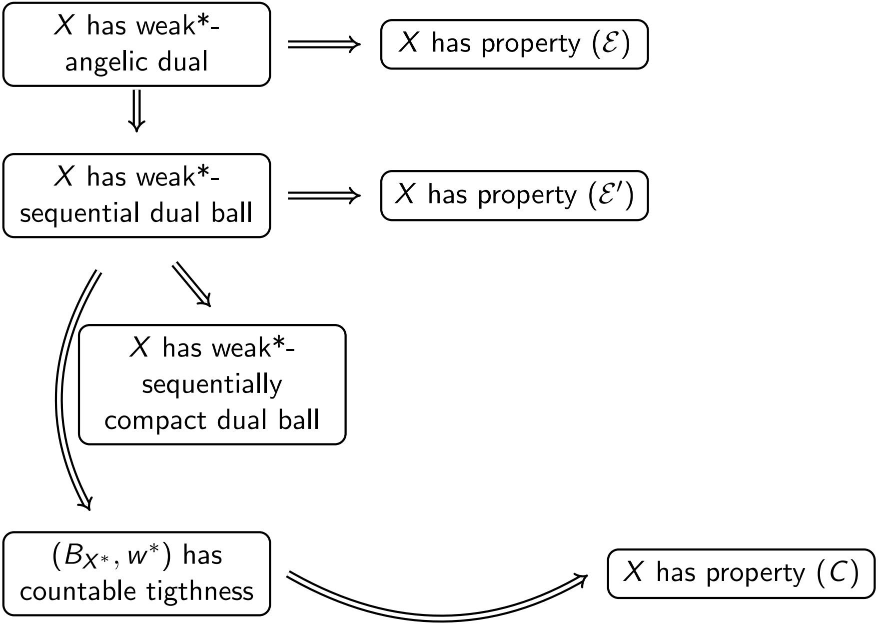
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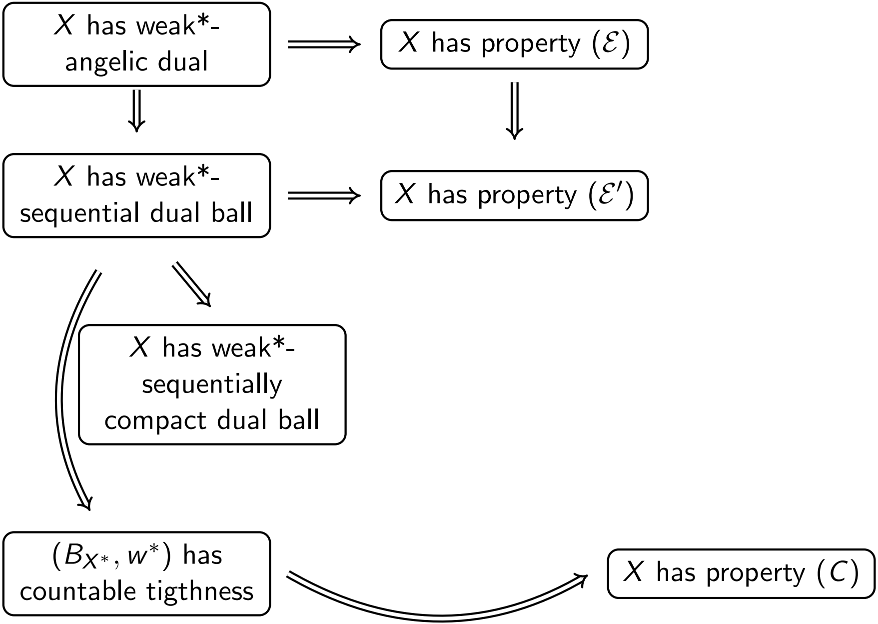
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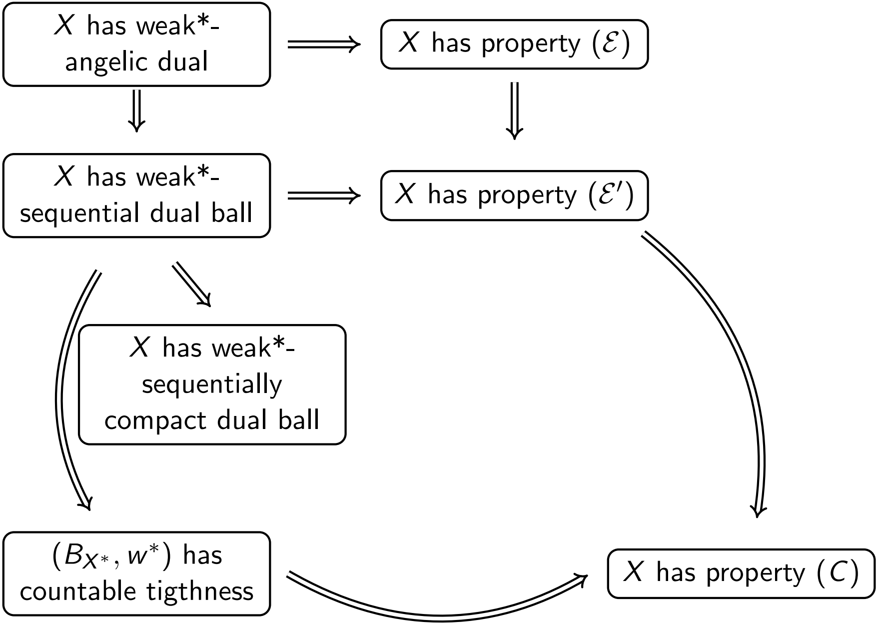

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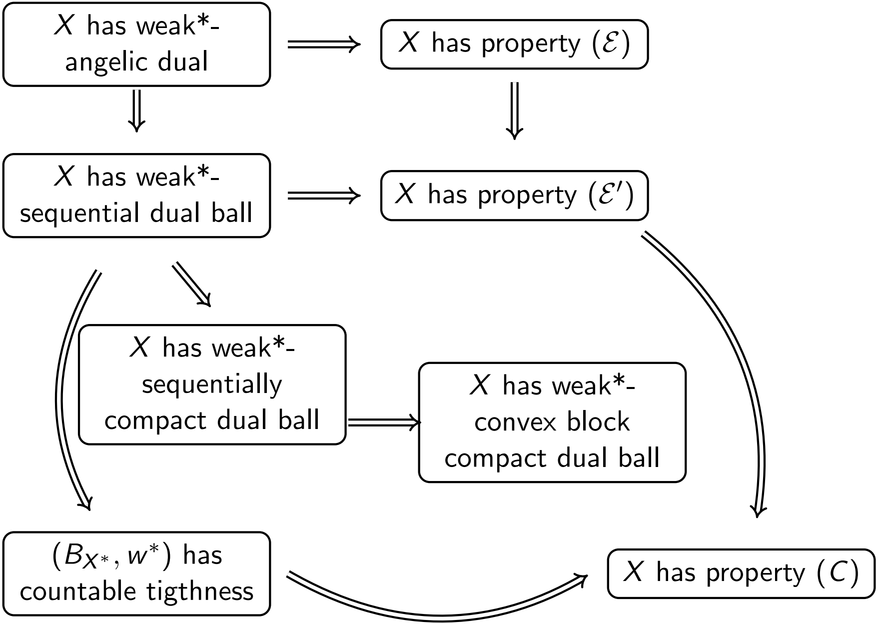
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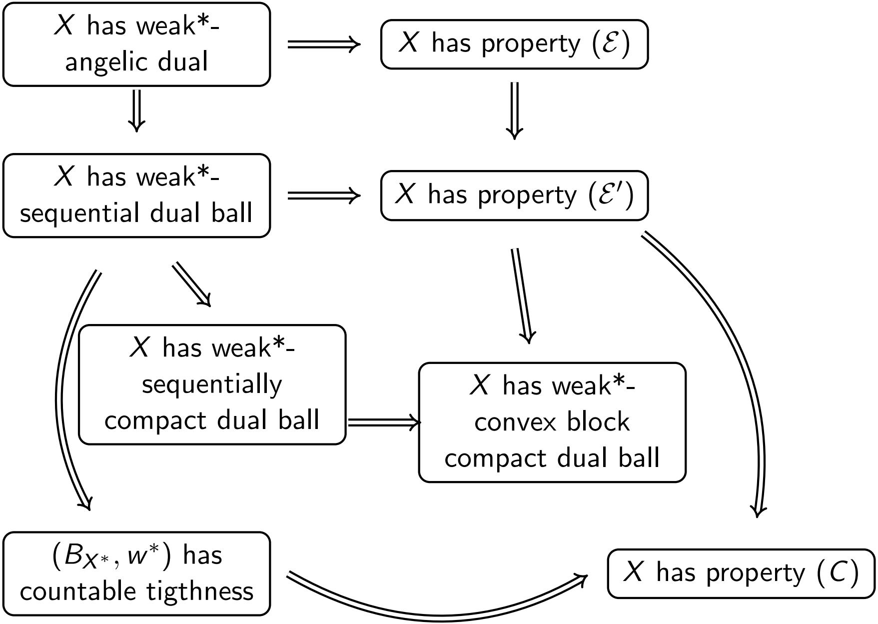

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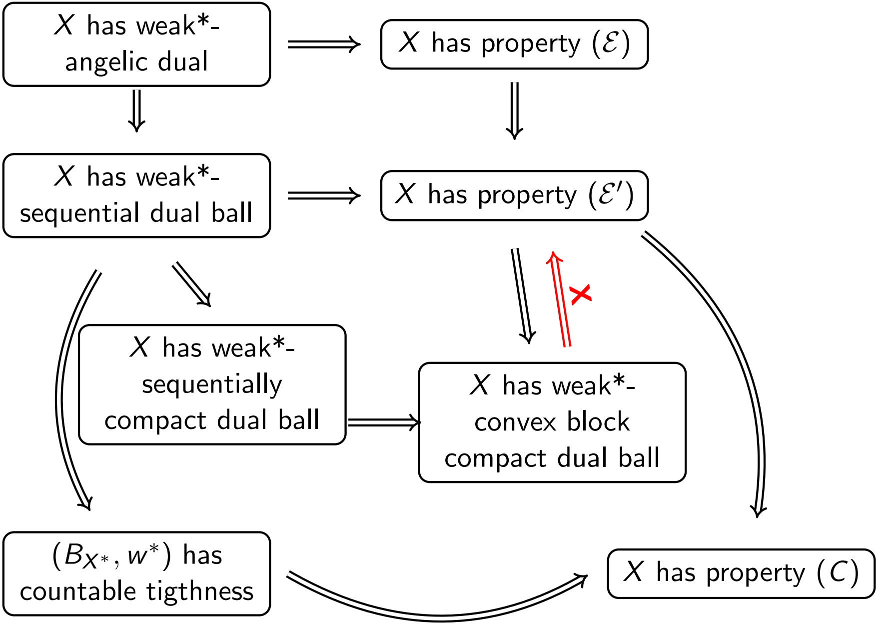

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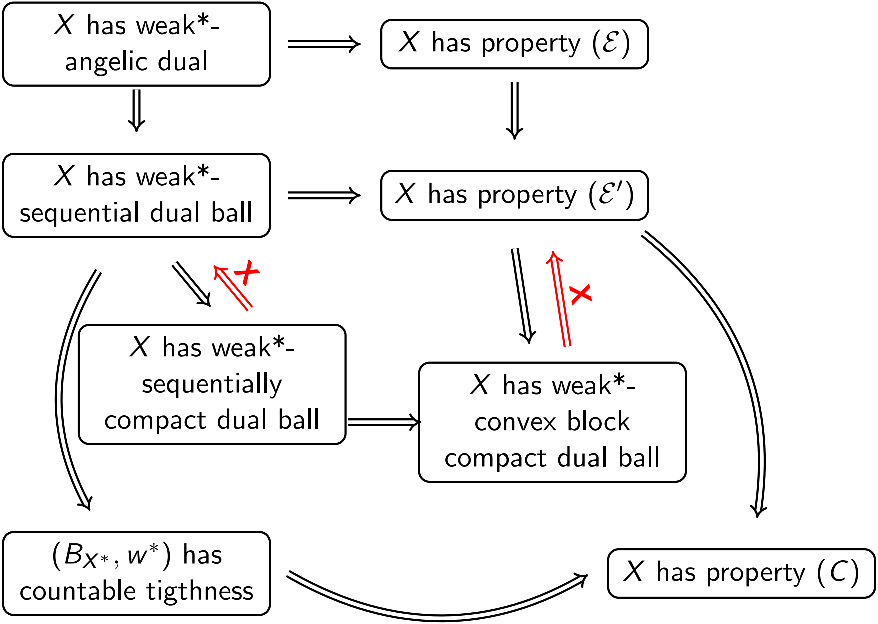
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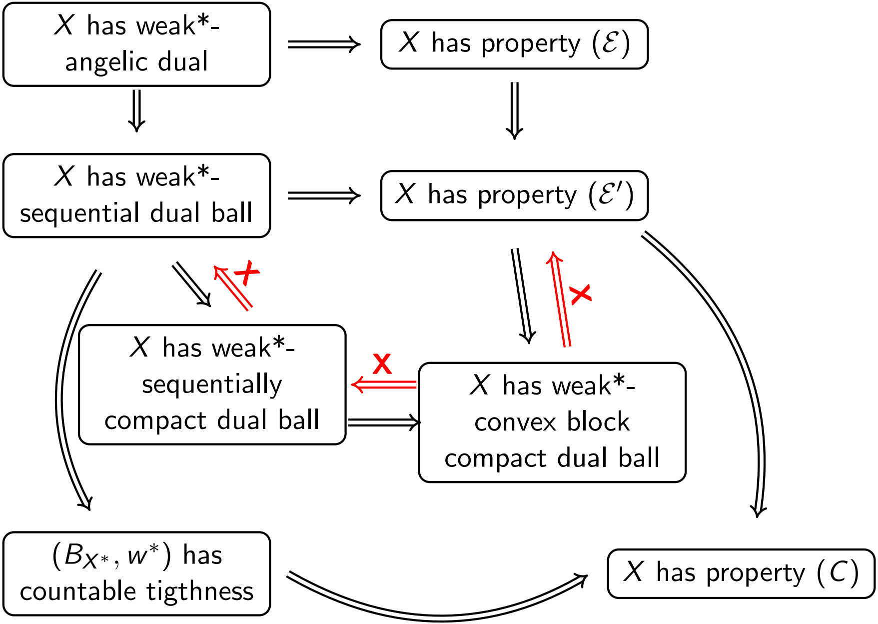
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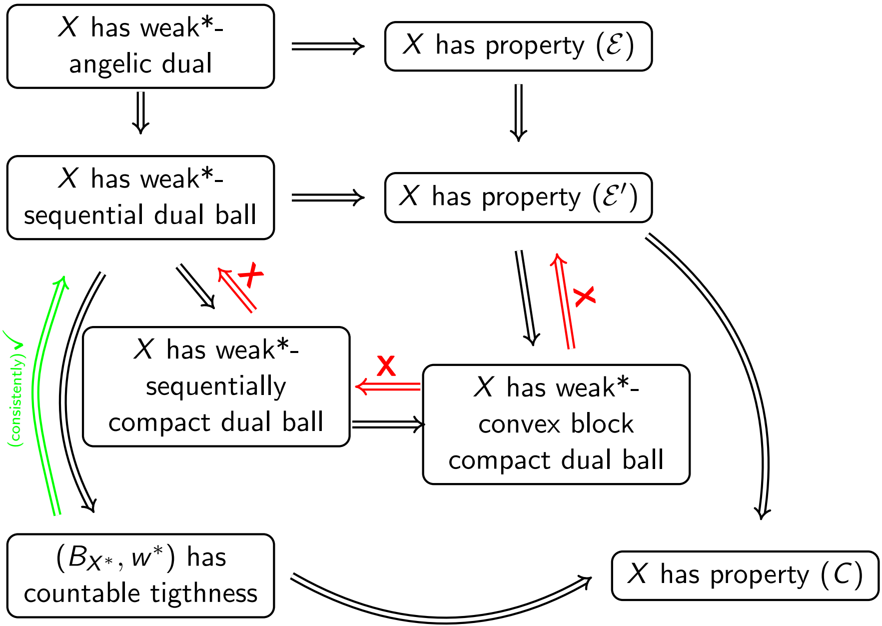

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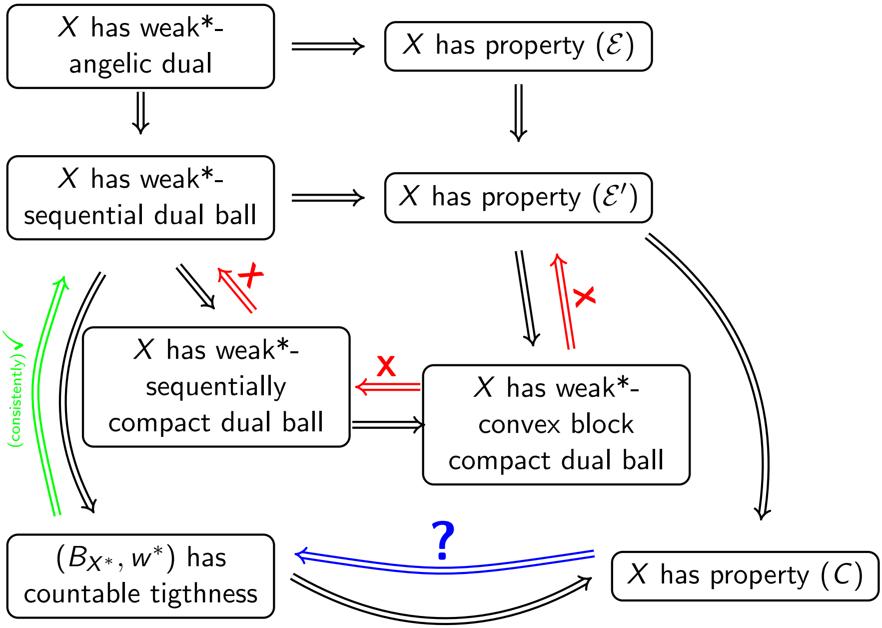
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Theorem (G.M.C., 2018)

If X is a Banach space with weak-sequentially compact dual ball and $Y \subseteq X$ is a subspace such that Y and X/Y have weak*-sequential dual ball, then X has weak*-sequential dual ball.*

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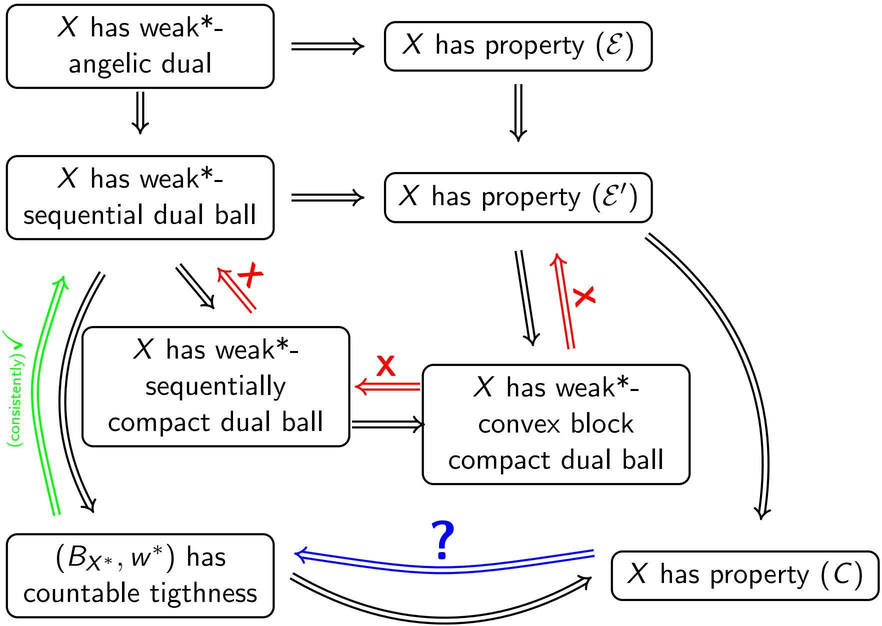
$$S_2 = S_1 \cup \{0\} = \overline{S_0}^{w^*}.$$

In particular, JL_2 does not have weak*-angelic dual.



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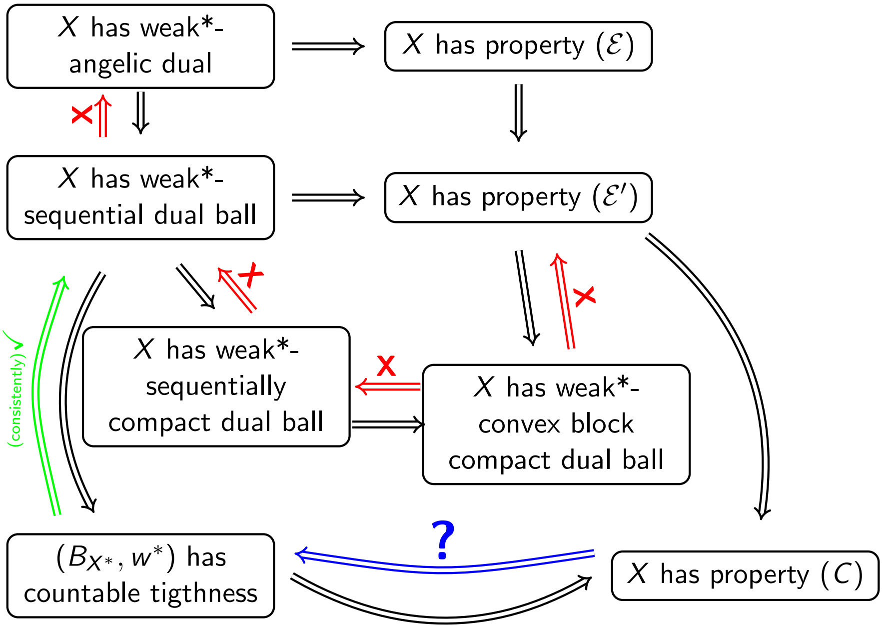

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Under CH, there exists a MAD family $\mathcal{F}^-$ for which the corresponding Johnson-Lindestrauss JL_2 **does not have** *property $(\mathcal{E})$.*

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$JL_2(\mathcal{F}^+)$ has the following property: every bounded sequence $(x_n^*)_{n\in\mathbb{N}}$ for which 0 is a weak*-cluster point has a subsequence $(x_{n_k}^*)_{k\in\mathbb{N}}$ such that $\lim_k \frac{1}{k}(x_{n_1}^* + \ldots + x_{n_k}^*) = 0$.

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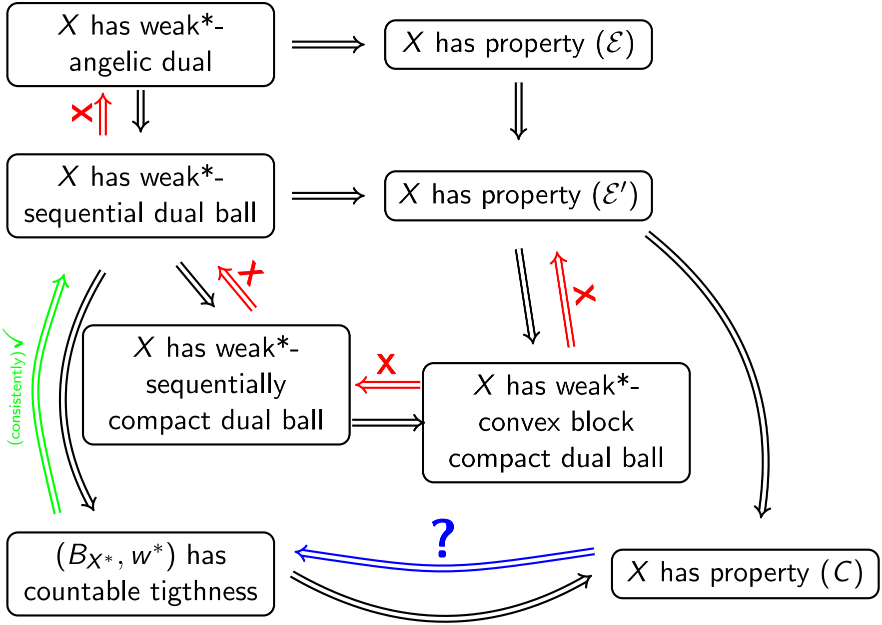
Under CH, after ω_1 steps no weak*-null sequence in $\mathrm{co}(\{e_n^* : n \in \mathbb{N}\})$ survives.



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X has property (ℰ′)
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X has weak*-convex block compact dual ball
(consistently)
(B_{X^*}, w^*) has countable tigthness
?
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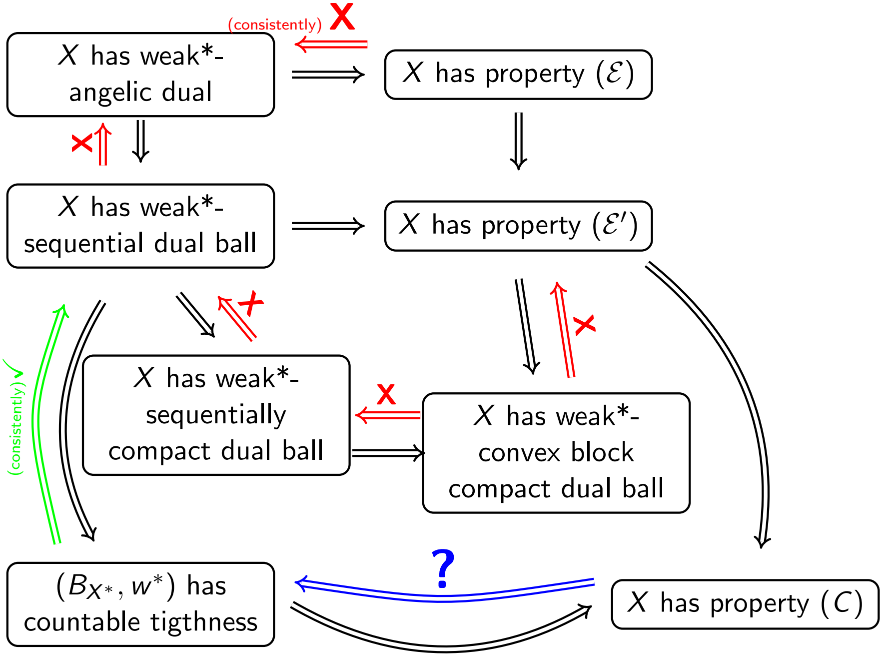

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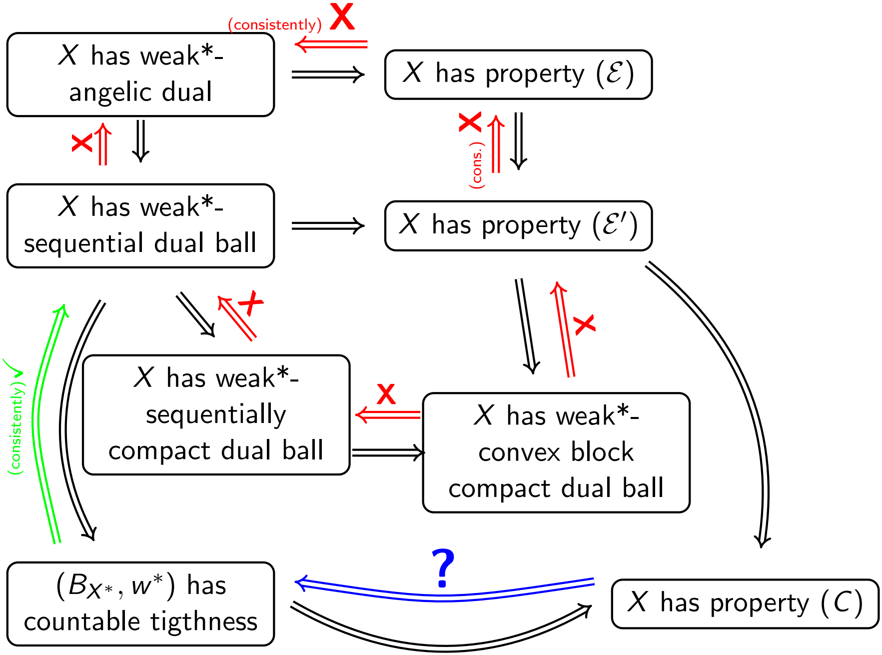
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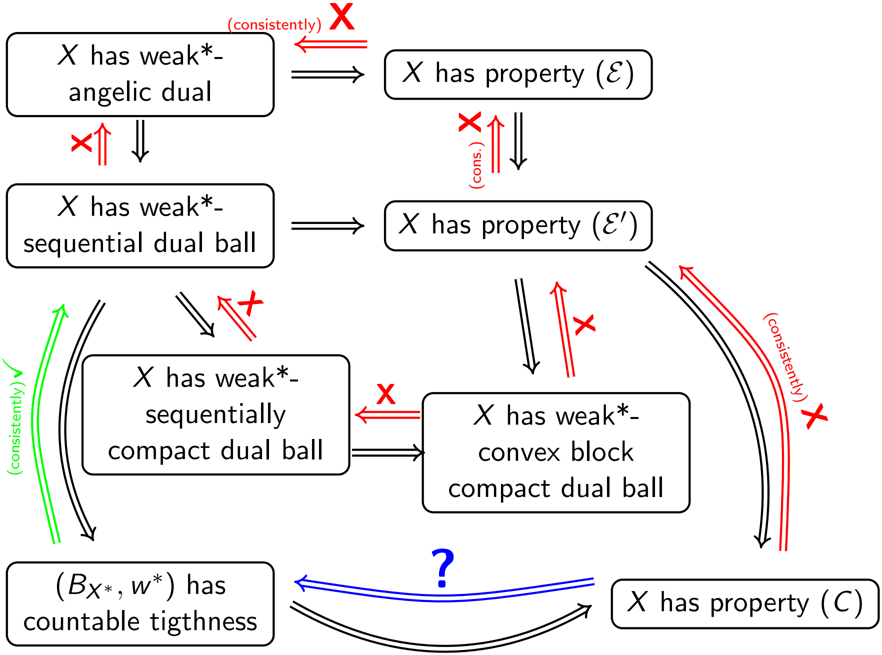

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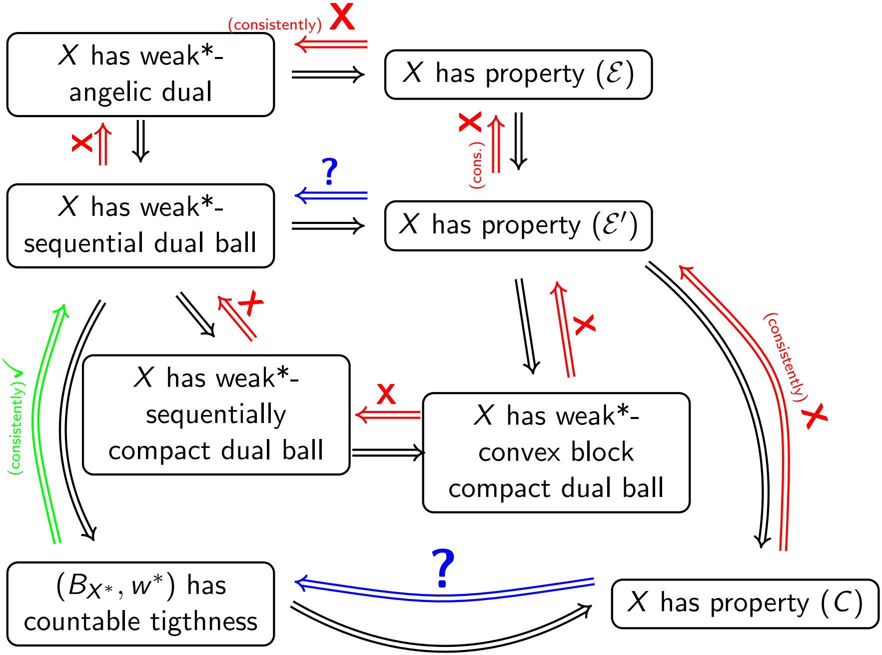

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Open Problem

Can $\mathcal{F}^+$ or $\mathcal{F}^-$ be constructed in ZFC without any extra set-theoretic assumption?

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