How can we use Fixed Point Theory in order to help ourshelf?

A joint work with Jesús Garcia-Falset and Simeon Reich

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Main aims

- To obtain an extension of Bolzano-Poincaré-Miranda theorem to **infinite** dimensional Banach spaces.
- To establish a result regarding the existence of periodic solutions to differential equations posed in an arbitrary Banach space.
- To prove an equivalence between our main result and Schauder and Brouwer theorems.

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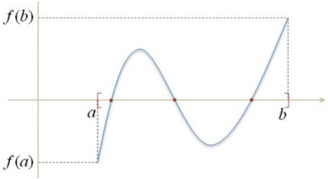
INTRODUCTION

x is a **zero** of a mapping $f : U \to \mathbb{B}$ if $f(x) = 0$.

Theorem 1 (Bolzano's theorem)

*If $f : [a, b] \to \mathbb{R}$ is a **continuous** function and $\boldsymbol{f(a)f(b) < 0}$, then there exists a point $x \in (a, b)$ such that $f(x) = 0$.*



Proof. (Bolzano in 1817 and Cauchy in 1821).

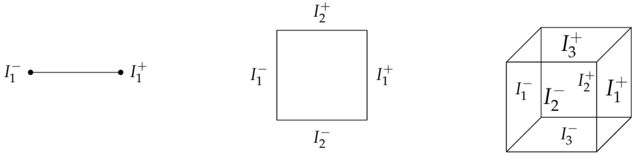
Theorem 2 (Poincaré-Miranda's theorem)

Let $P := \{x \in \mathbb{R}^n : |x_i| \leqslant L \text{ for all } 1 \leqslant i \leqslant n\}$*. Suppose that* $F = (f_1, \cdots, f_n) : P \to \mathbb{R}^n$ *is a* ***continuous*** *mapping on P such that*

(a) $f_i(x_1, \cdots, x_{i-1}, -L, x_{i+1}, \cdots, x_n) > 0$ *for* $1 \leqslant i \leqslant n$,

(b) $f_i(x_1, \cdots, x_{i-1}, L, x_{i+1}, \cdots, x_n) < 0$ *for* $1 \leqslant i \leqslant n$.

Then there exists at least one point $x \in P$ *such that* $F(x) = 0$.



Proof. Poincaré (1883) without proof. The first proof by Miranda (1940).

Notations and Preliminaries

$(\mathbb{B}, \|\cdot\|)$ a Banach space and $\mathbb{B}^* := \mathcal{L}(\mathbb{B}, \mathbb{R})$ its dual space.

For $C \subseteq \mathbb{B}$, $\overline{C}$, ∂C, $\mathrm{int}(C)$ and $\mathrm{conv}(C)$ the closure, the boundary, the interior and the convex hull of C, respectively.

$B_r[x]$ and $S_r[x]$ the closed ball and the sphere of center x and radius r, respectively.

the normalized duality mapping J at x, is defined as

$$J(x) = \{j \in \mathbb{B}^* : \langle x, j\rangle := j(x) = \|x\|^2, \ \|j\| = \|x\|\}.$$

We introduce a class of functionals $[\![\cdot,\cdot]\!] : \mathbb{B} \times \mathbb{B} \to \mathbb{R}$ satisfying:

(C_1) $[\![x,x]\!] > 0$ for all $x \in \mathbb{B}$ with $x \neq 0$.

(C_2) $[\![\lambda x, x]\!] = \lambda [\![x,x]\!]$ for all $x \in \mathbb{B}$ and $\lambda \in \mathbb{R}$.

Example 1

(1) If $\mathbb{B}$ is a pre-Hilbert space then $[\![\cdot,\cdot]\!]$ is its inner-product.

(2) In general, if $\mathbb{B}$ is a Banach space, we may define $[\![\cdot,\cdot]\!] : \mathbb{B} \times \mathbb{B} \to \mathbb{R}$ as either

$$[\![x,y]\!] = \langle x,y\rangle_+ := \sup_{j(y)\in J(y)} \langle x, j(y)\rangle = \max_{j(y)\in J(y)} \langle x, j(y)\rangle$$

or

$$[\![x,y]\!] = \langle x,y\rangle_- := \inf_{j(y)\in J(y)} \langle x, j(y)\rangle = \min_{j(y)\in J(y)} \langle x, j(y)\rangle.$$

In particular, if $\mathbb{B}$ is a smooth Banach space, then $[\![\cdot,\cdot]\!]$ can be given by

$$[\![x,y]\!] = \langle x, J(y)\rangle.$$

Our main result

$f : U \subseteq \mathbb{B} \to \mathbb{B}$ is **completely continuous** if f is continuous and $f(C)$ is relative compact for all C bounded subset of U.

Theorem 3

*Let U be a **bounded closed** subset of a Banach space $\mathbb{B}$ with int$(U) \neq \emptyset$, and $[\![\cdot,\cdot]\!] : \mathbb{B} \times \mathbb{B} \to \mathbb{R}$ be a functional satisfying*

(C_1) $[\![x,x]\!] > 0$ *for all* $x \in \mathbb{B}$ *with* $x \neq 0$.

(C_2) $[\![\lambda x,x]\!] = \lambda [\![x,x]\!]$ *for all* $x \in \mathbb{B}$ *and* $\lambda \in \mathbb{R}$.

*If $f : U \to \mathbb{B}$ is a **completely continuous** mapping and there exists $z \in \mathrm{int}(U)$ such that $[\![f(x), x - z]\!]$ **has constant sign for all** $x \in \partial U$, then $0 \in \overline{f(U)}$. Moreover, if $0 \notin \partial f(U) \backslash f(U)$, then $0 \in f(U)$.*

SOME CONSEQUENCES

Corollary 1

Let $\mathbb{B}$ *be a Banach space with* $[\![\cdot,\cdot]\!] : \mathbb{B}\times\mathbb{B}\to\mathbb{R}$ *a functional satisfying* (C_1) *and* (C_2), *and* $z\in X$. *If* $f : B_r[z]\to\mathbb{B}$ *is a completely continuous mapping such that* $[\![f(x), x-z]\!]$ *has constant sign for all* $x\in S_r(z)$, *then* $0\in\overline{f(B_r[z])}$. *Moreover, if* $\mathbb{B}$ *is reflexive and* f *is weak-strong continuous on* $B_r[z]$, *then* $0\in f(B_r[z])$.

Corollary 2

Let U *be a nonempty bounded and closed subset of a Banach space* $\mathbb{B}$ *with* $\operatorname{int}(U)\neq\emptyset$. *If* $f : U\to\mathbb{B}$ *is a completely continuous mapping and there exists* $z\in \operatorname{int}(U)$ *such that either* $f(x)\notin\{\lambda(x-z) : \lambda<0\}$ *for all* $x\in\partial U$ *or* $f(x)\notin\{\lambda(x-z) : \lambda>0\}$ *for all* $x\in\partial U$, *then* $0\in\overline{f(U)}$.

Remark

As a consequence of the previous result, we obtain

- *Proposition 4 in* ***Alefeld, Frommer, Heindl, Mayer,*** *On the existence theorems of Kantorovich, Miranda and Borsuk,* ***(2004)****.*
- *Theorem 1 in* ***C.H. Morales,*** *A Bolzano's theorem in the new millennium****, (2002)****.*
- *Corollary 3 in* ***Isac,*** *Some solvability theorems for nonlinear equations with applications to projected dynamical systems****, (2009)****.*
- *A generalization of Poincaré-Bohl's theorem given in* ***Fonda, Gidoni,*** *Generalizing the Poincaré-Miranda Theorem: The avoiding cones condition****, (2016)****.*

Consequences in finite dimensional spaces

A subset $D \subset \mathbb{R}^n$ is said to be a **convex body** if it is a compact convex set with nonempty interior.

Given a point $x \in \partial D$, we define the **normal cone** to $\operatorname{int}(D)$ in x as

$$\mathcal{N}_D(x) := \{ v \in \mathbb{R}^n : \langle v, y - x \rangle < 0, \text{ for every } y \in \operatorname{int}(D) \}$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product in $\mathbb{R}^n$.

Corollary 3

Let D be a ***convex body*** *in $\mathbb{R}^n$ endowed with an arbitrary norm $\| \cdot \|$. Let $f : D \to \mathbb{R}^n$ be a* ***continuous function*** *and assume that for all $x \in \partial D$ there exists $a(x) \in \mathcal{N}_D(x)$ such that $\langle f(x), a(x) \rangle \geqslant 0$, then f has a zero in D.*

Proof of Bolzano's Theorem.

$\mathbb{B} = \mathbb{R}$ with $\|\cdot\| = |\cdot|$.

Taking $z = \frac{a+b}{2}$ and $r = \frac{b-a}{2}$, we have $B_r[z] = [a, b]$ and $S_r(z) = \{a, b\}$.

We define $[\![x, y]\!] = x\,y$.

The hypothesis $f(a)f(b) < 0$, implies that $[\![f(x), x - z]\!] = f(x)(x - z)$ has a constant sign for all $x \in S_r(z)$.

Since $(\mathbb{R}, |\cdot|)$ is finite dimensional Banach space, we may apply Corollary 1 to obtain the result. □

Proof of Bolzano-Poincaré-Miranda Theorem.

$$P := \{x \in \mathbb{R}^n : \ |x_i| \leqslant L \text{ for all } 1 \leqslant i \leqslant n\}$$

is a convex body in $\mathbb{R}^n$ and

$$\partial P = \{x \in P : \ x_i = \pm L \text{ for some } 1 \leqslant i \leqslant n\}.$$

If $x \in \partial P$, then $x = (x_1, \cdots, x_{i-1}, \pm L, x_{i+1}, \cdots, x_n)$.

In this case, we take $a(x) = (0, \ldots, 0, \pm L, 0, \ldots, 0) \in \mathcal{N}_P(x)$.

Consider the function $g := -f$. Bearing in mind conditions (a) and (b), we obtain that

$$\langle g(x), a(x) \rangle = -f_i(x_1, \cdots, x_{i-1}, \pm L, x_{i+1}, \cdots, x_n)(\pm L) > 0.$$

The above argument says that every conditions in Corollary 3 is satisfied, which allows us to obtain the result. □

Consequences in infinite dimensional spaces

We can use our theorem in order to get several results in infinite dimensional spaces.

An application to systems of nonlinear equations

Theorem 4

Let L be a ***linear homeomorphism*** *from* $(\mathbb{R}^n, \|\cdot\|)$ *into itself such that* $\ell := \min\left\{ \|L(x)\| : \|x\| = 1\right\} > 0$. *Let* $g : \mathbb{R}^n \to \mathbb{R}^n$ *be a* ***continuous*** *mapping. If there exists* $R > 0$ *such that* $\|g(x)\| \leqslant \ell R$ *for all* $x \in B_R[0]$, *then the nonlinear equation* $L(x) + g(x) = 0$ *has at least one solution in* $B_R[0]$.

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An application to systems of nonlinear equations

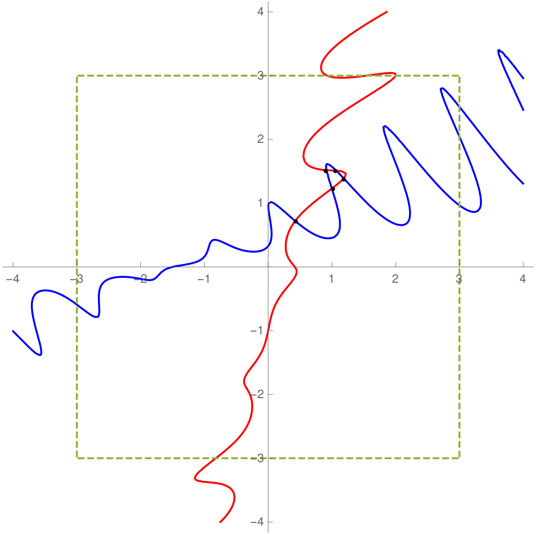
Example 2

The system of the nonlinear equations

$$(S)\begin{cases} -2x+7y+4y\cos(5x+3y)=3 \\ 7x-2y-3x\sin(2x-5y)=2 \end{cases}$$

has at least a solution in $[-3,3]\times[-3,3]$.

An application to systems of nonlinear equations



An application to differential equations

We can use our result in order to prove the existence of solutions for the following second order differential equation:

$$\begin{cases} u''(t) = f(t, u(t)) + h(t) \\ u(a) = u(b), \\ u'(a) = u'(b), \end{cases}$$

where $\mathbb{B}$ is a reflexive real Banach space, $f : [a, b] \times \mathbb{B} \to \mathbb{B}$ is a sequentially weak-strong continuous mapping and $h : [a, b] \to \mathbb{B}$ is a continuous function on $\mathbb{B}$.

Example 3

If $h_1, \dots, h_n : [a,b] \to \mathbb{R}$ are continuous functions with $\int_a^b h_i(t)\,dt = 0$, then the problem

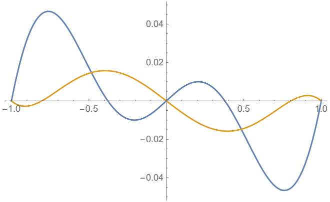
$$\begin{cases} u_1''(t) = \dfrac{u_1(t)}{1 + |u_1(t)| + \cdots + |u_n(t)|} + h_1(t) \\ \qquad \vdots \\ u_n''(t) = \dfrac{u_n(t)}{1 + |u_1(t)| + \cdots + |u_n(t)|} + h_n(t) \\ u_1(a) = u_1(b), \dots, u_n(a) = u_n(b), \\ u_n'(a) = u_n'(b), \dots, u_n'(a) = u_n'(b), \end{cases}$$

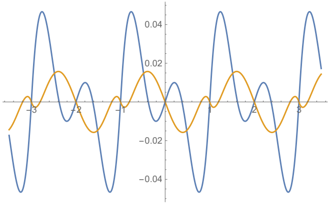
has at least a solution in $\mathcal{C}^2([a,b], \mathbb{R}^n)$.

An application to differential equations

$$\begin{cases} u_1''(t) = \dfrac{u_1(t)}{1 + |u_1(t)| + |u_2(t)|} + t - \sin(5t) \\ u_2''(t) = \dfrac{u_2(t)}{1 + |u_1(t)| + |u_2(t)|} + t - 2t^3 \\ u_1(-1) = u_1(1), u_n(-1) = u_2(1), \\ u_1'(-1) = u_1'(1), u_2'(-1) = u_2'(1), \end{cases}$$

has at least a solution in $\mathcal{C}^2([-1,1], \mathbb{R}^2)$





An application to differential equations

Example 4

Let $h_1, \dots, h_n : [a,b] \to \mathbb{R}$ be continuous functions with $\int_a^b h_i(t)\,dt = 0$ and let $0 < \alpha < 1$. Then, the problem

$$\begin{cases} u_1''(t) = \dfrac{u_1(t)\sqrt{(u_1^2(t) + \cdots + u_n^2(t))^\alpha}}{1 + \sqrt{u_1^2(t) + \cdots + u_n^2(t)}} + h_1(t) \\ \qquad \vdots \\ u_n''(t) = \dfrac{u_n(t)\sqrt{(u_1^2(t) + \cdots + u_n^2(t))^\alpha}}{1 + \sqrt{u_1^2(t) + \cdots + u_n^2(t)}} + h_n(t) \\ u_1(a) = u_1(b), \dots, u_n(a) = u_n(b), \\ u_1'(a) = u_1'(b), \dots, u_n'(a) = u_n'(b), \end{cases}$$

has at least a solution in $\mathcal{C}^2([a,b], \mathbb{R}^n)$.

Example 5

Let Ω be an open convex and bounded subset of $\mathbb{R}^n$ and consider $\phi, \rho : [a,b] \times \Omega \to \mathbb{R}$ two functions. We can prove the existence of solutions for the following partial differential equation:

$$\begin{cases} \displaystyle\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \left(\frac{\partial^2 \phi}{\partial t^2}(t,x) - \rho(t,x) \right) = \phi(t,x) \\ \phi(a,x) - \phi(b,x) = \dfrac{\partial \phi}{\partial t}(a,x) - \dfrac{\partial \phi}{\partial t}(b,x) = 0, \textit{for all } x \in \Omega \\ \dfrac{\partial^2 \phi}{\partial t^2}(t,x) - \rho(t,x) = 0 \textit{ for all } (t,x) \in (a,b) \times \partial\Omega. \end{cases}$$

A final remark

Theorem 5

The following theorems are equivalent:

(*a*) *Brouwer's fixed point theorem.*

(*b*) *Schauder's fixed point theorem.*

(*c*) *Theorem 3.*

References

Every result on this talk can be found in the following papers and the references given there.

D. Ariza-Ruiz, J. García-Falset, S. Reich.
The Bolzano-Poincaré-Miranda theorem in infinite dimensional Banach spaces.
J. Fixed Point Theory Appl. **21**, 59 (2019).

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Periodic solutions to second order nonlinear differential equations in Banach spaces.
(*preprint*).

Thank you

for your attention!