Open problems session

V CONGRESO DE JÓVENES INVESTIGADORES DE LA RSME
Functional analysis session
27-31 January 2020, Castellón, Spain

Problem (Abraham Rueda – 2 (big size) pizzas).

Given a Banach space X, we define the *(N-fold) projective symmetric tensor product* of X, denoted by $\widehat{\otimes}_{\pi,s,N}X$, as the completion of the space $\otimes^{s,N}X$ under the norm

$$\|u\| := \inf\left\{\sum_{i=1}^{n}|\lambda_i|\|x_i\|^N : u := \sum_{i=1}^{n}\lambda_i x_i^N, n\in\mathbb{N}, x_i\in X\right\}.$$

The dual, $(\widehat{\otimes}_{\pi,s,N}X)^* = \mathcal{P}(^N X)$, is the Banach space of N-homogeneous continuous polynomials on X, and notice that $B_{\widehat{\otimes}_{\pi,s,N}X} = \overline{\text{co}}(\{x^N : x\in S_X\})$.

Let $\{x_n\}$ be a sequence in S_X which is isometric to the c_0-basis.
Is $\{x_n^N\}\subseteq S_{\widehat{\otimes}_{\pi,s,N}X}$ a c_0-basis in $widehat\otimes_{\pi,s,N}X$?

In the context of non-symmetric tensor product is it known that the product of two c_0 basis is a c_0 basis in the projective tensor product [3, Section 2.5].

Furthermore, the answer is positive for $N=2$ (see [2, Remark 2.4]).

A positive answer to the previous problem may motivate a positive answer to [1, Question 4.3].

References

[1] J. Becerra Guerrero, G. López-Pérez and A. Rueda Zoca, *Some results on almost square Banach spaces*, J. Math. Anal. Appl. **438** (2016), 1030–1040.

[2] A. Rueda Zoca, *Almost squareness and strong diameter two property in tensor product spaces*, RACSAM **114**, 84 (2020). https://doi.org/10.1007/s13398-020-00816-4

[3] R. A. Ryan, *Introduction to tensor products of Banach spaces*, Springer Monographs in Mathematics, Springer-Verlag, London, 2002.

Problem (Abraham Rueda – One litre of high-quality olive oil from Jaén).

Given two Banach spaces X and Y we will denote by $L(X,Y)$ (resp. $K(X,Y)$) the space of all bounded (resp. compact) linear operators from X to Y. Recall that the *projective tensor product* of X and Y, denoted by $X\widehat{\otimes}_\pi Y$, is the completion of $X\otimes Y$ under the norm given by

$$\|u\| := \inf\left\{\sum_{i=1}^n \|x_i\|\|y_i\| : u = \sum_{i=1}^n x_i \otimes y_i\right\}.$$

It is known that $B_{X\widehat{\otimes}_\pi Y} = \overline{\text{co}}(B_X\otimes B_Y) = \overline{\text{co}}(S_X\otimes S_Y)$ [3, Proposition 2.2]. Moreover, given Banach spaces X and Y, it is well known that $(X\widehat{\otimes}_\pi Y)^* = L(X,Y^*)$ (see [3] for background).

Given X, Y two Banach spaces so that X^{**} and Y^{**} have the metric approximation property then $X\widehat{\otimes}_\pi Y \subseteq X^{**}\widehat{\otimes}_\pi Y^{**} \subseteq (X\widehat{\otimes}_\pi Y)^{**}$, where the previous inclusions are isometric.

Assume that $K(X,Y^*) \subsetneq L(X,Y^*)$. When $B_{X^{**}}\otimes B_{Y^{**}} = \{x^{**}\otimes y^{**} : x^{**}\in B_{X^{**}}, y^{**}\in B_{Y^{**}}\}$ is weak-star closed in $(X\widehat{\otimes}_\pi Y)^{**}$.

References

[1] R. A. Ryan, *Introduction to tensor products of Banach spaces*, Springer Monographs in Mathematics, Springer-Verlag, London, 2002.

Problem (Abraham Rueda – A dinner and an extra calimocho.

Recall that a Banach space X has the:

1. *strong diameter two property (SD2P)* if every convex combination of slices of B_X has diameter exactly two.

2. *diametral strong diameter two property (DSD2P)* if, for every convex combination of slices C of B_X, every $x \in C$ and every $\varepsilon > 0$ there exists $y \in C$ such that

$$\|x - y\| > 1 + \|x\| - \varepsilon.$$

3. *Daugavet property (DP)* if, for every convex combination of slices C of B_X, every $x \in B_X$ and every $\varepsilon > 0$ there exists $y \in C$ such that

$$\|x - y\| > 1 + \|x\| - \varepsilon.$$

Note that the definition of the Daugavet property is a reformulation of the original property which follows from the proof of [4, Lemma 2.2].

It is known that DP implies the DSD2P, but the converse is unknown [1, Question 4.1]. Also clearly the DSD2P implies the SD2P.

From the results of [2] it follows that for an infinite set I of cardinality strictly bigger than the density character of $L_1([0,1])^{**}$ and for $1 < p < 2$ it follows that $L_\infty([0,1])\widehat{\otimes}_\pi \ell_p(I)$ has the SD2P but fails the Daugavet property (since its predual fails the Daugavet property). Notice that $L_\infty([0,1])$ has the DP.

Does $L_\infty([0,1])\widehat{\otimes}_\pi \ell_p(I)$ have the DSD2P?

References

[1] J. Becerra Guerrero, G. López-Pérez and A. Rueda Zoca, *Diametral diameter two properties in Banach spaces*, J. Conv. Anal. **25** (2018), 817–840.

[2] G. López-Pérez and A. Rueda Zoca, *L-orthogonality, octahedrality and the Daugavet property in Banach spaces*, preprint. Available at arXiv.org with reference arXiv:1912.09039.

[3] R. A. Ryan, *Introduction to tensor products of Banach spaces*, Springer Monographs in Mathematics, Springer-Verlag, London, 2002.

[4] R. V. Shvidkoy, *Geometric aspects of the Daugavet property*, J. Funct. Anal. **176**, 2 (2000), 198-212.

Problem (Sheldon Dantas – A free tour in Prague).

The Bishop-Phelps-Bollobás property for operators (BPBp, for short) can be also considered just for compact operators, where all the involved operators as compact ones, that is, given $\epsilon > 0$, there exists $\eta(\epsilon) > 0$ such that whenever T is a compact operator with norm one and x is an element in the sphere of X satisfying

$$\|T(x)\| > 1 - \eta(\epsilon),$$

there are a new norm one compact operator S and a new norm one element x_0 such that

$$\|S(x_0)\| = 1, \|x_0 - x\| < \epsilon, and \|S - T\| < \epsilon.$$

In this case, we say that the pair of Banach spaces (X, Y) satisfies the Bishop-Phelps-Bollobás property for compact operators (BPBp-K, for short).

It is known that the pair $(L_1[0,1], C[0,1])$ has the BPBp-K (B. Cascales, J. Guirao, and V. Kadets, 2013) but not the BPBp because the set $NA(L_1[0,1], C[0,1])$ of all operators that attain the norm from $L_1[0,1]$ into $C[0,1]$ is not dense in the whole space of bounded linear operators from $L_1[0,1]$ into $C[0,1]$ (W. Schachermayer, 1983). This means that the BPBp-K cannot imply the BPBp. We do not know if the BPBp implies the BPBp-K.

Problem (Tommaso Russo – 5 Czech (big) beers for each problem).

1. Let H be a dense hyperplane in ℓ_1. Does H admit a C^1-smooth norm?
2. Let Y be the normed space of simple functions in $L_1[0,1]$. Does Y admit a C^1-smooth norm?
3. Let Y be the linear span of the canonical basis of $\ell_1(\Gamma)$, where $|\Gamma| \geq \mathfrak{c}^+$. Does Y admit an analytic norm?
4. (Bonus, not asked at the Open Problem session.) Let X be a Banach space. Does B_X contain a 1-separated set of cardinality $\operatorname{dens} X$?
   Comment: A subset A of B_X is 1-*separated* if $\|x-y\| \geq 1$ for distinct $x, y \in A$. The answer is YES for separable, or reflexive Banach spaces (Riesz' lemma). It is also positive if $\operatorname{dens} X = w^*\text{-}\operatorname{dens} X^*$ (Mazur's lemma), and when X is a $C(K)$ space (Urysohn's lemma).

Problem (Javier Falcó – One Big pizza + beer for each problem).

- It is known that the set of 2-homogenenous polynomials whose extension to the bidual attain their norm is norm dense in the space of 2-homogenenous polynomials. Does the result holds for $n \geq 3$?
- Are the set of linear and continuous norm attaining operator from X to $\mathbb{R}^2$ dense in the space of linear and continuous $\mathcal{L}(X;\mathbb{R}^2)$ for every Banach space X?

References

[1] Aron, Richard M.; García, Domingo; Maestre, Manuel On norm attaining polynomials. Publ. Res. Inst. Math. Sci. 39 (2003), no. 1, 165–172.